Combinatorics and Probability

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Combinatorics is a fancy word for counting. Probability is a fancy word for chance. The two are intimately related, as we will discover.

1 Permutations

A permutation is an arrangement of objects in which order matters. We typically want to count how many possible permutations there are. Let's start with a fundamental example.

1. How many four-letter words can be made from the letters of the word *math*, using each letter once per word? The words don't have to mean anything. Similarly (Exeter 89:9) how many nine-letter words can be formed from the letters of *hyperbola*?

An extremely useful notation when dealing with permutations is $n!$, read "n factorial". The definition is $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$, so for example $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$.

2. Express your answers to problem 1 using factorial notation. Now determine how many four-letter words can be made from the letters of *hyperbola*, how many three-letter words can be made from *hyperbola*, and how many three-letter words can be made from *math*. Try to express your answer both with and without factorial notation.

3. In general, given n distinct letters, how many words of length r can be formed? Express your formula both with and without factorial notation. What happens when $r = 0$? What happens when $r > n$?

The notation for the number of permutations of r objects taken from a collection of n distinct objects is ${}_nP_r$, which is read "n permute r". If you solved problem 3, you discovered a formula for ${}_nP_r$.

4. How many distinct four-letter words can be made from the letters of *gala*, using each letter once per word? Write out all of the possible words. How does this differ from *math*? Write your answer using factorial notation if you can.

5. How many distinct six-letter words can be made from *common*? Write out the possibilities if you are having trouble. Can you see the pattern? Write the answer using factorial notation.

6. How many distinct eleven-letter words can be made from *Mississippi*? Write the answer using factorial notation. How many words the same length as your name can be made from the letters of your first name? If that's too easy try out your full name.

2 Combinations

A combination is an arrangement of objects in which order does not matter. Again we typically want to count how many possible combinations there are.

7. How many ways are there to choose four letters from the word *math*, where now we don't care about order—we are just choosing four letters? I'll tell you the answer: It is one. Do you understand why? Now how many ways are there to choose three letters from *math*, where again we don't care about order? How many ways are there to choose two letters from *math*? One letter? Zero letters? Compare your answers to the case of permutations, in which order matters. See if you can write your answers in factorial notation.

8. Repeat problem 7 using the word *hyperbola*. Find out how many ways you can choose 9 letters, then 8, then so on down to zero. Again try to write your answers in factorial notation. What patterns and symmetry do you see?

9. In general, given n distinct letters, how many ways are there to choose r of them? Express your formula both with and without factorial notation. What happens when $r = 0$? What happens when $r > n$?

One notation for the number of combination of r objects taken from a collection of n distinct objects is ${}_nC_r$, which is read "n choose r". However the much more common notation for combinations is $\binom{n}{r}$, which is also read "n choose r" and which is the notation you should use. Combinations are much more common and important than permutations, so they have this special notation. If you solved problem 9, you discovered a formula for $\binom{n}{r}$.

3 Combinatorics

Now that you understand the basics of permutations and combinations, you can practice and also start to learn some more sophisticated counting techniques.

10. Exeter 86:6.

11. Exeter 86:7.

12. (2005 AMC 10A, problem 14). How many three-digit numbers satisfy the property that the middle digit is the average of the first and last digits?

13. Exeter 88:5.

14. Exeter 92:9.

15. (2008, AMC 10B, problem 21). Ten chairs are spaced evenly around a round table and numbered clockwise from 1 through 10. Five married couples are to sit in the chairs with men and women alternating, and no one is to sit either next to or directly across from his or her spouse. How many seating arrangements are possible?

16. Exeter 93:7.

17. (2009 AMC 10B, problem 11). How many 7 digit palindromes (numbers that read the same backward and forward) can be formed using the digits 2, 2, 3, 3, 5, 5, 5?

18. (2010 AMC 10A, problem 22). Eight points are chosen on a circle, and chords are drawn connecting every pair of points. No three chords intersect in a single point inside the circle. How many triangles with all three vertices in the interior of the circle are created?

4 Probability

Basic probability is simply counting. For example say you flip a coin. There are two possible outcomes: heads or tails. If we care about getting heads, there is only one possibility, so the probability of getting heads is one out of two, or $\frac{1}{2}$. Note that we are assuming each outcome is equally likely, something we will assume unless otherwise specified.

Similarly say you have a deck of 52 cards. If you pick one card the probability of getting the queen of hearts is $\frac{1}{52}$. The probability if getting any queen is $\frac{4}{52} = \frac{1}{13}$, since there are four queens. The probability of getting any heart is $\frac{13}{52} = \frac{1}{4}$, since there are 13 hearts. The probability of getting a red queen is $\frac{2}{52} = \frac{1}{26}$, and so forth. The idea should be clear: To calculate the probability, divide the number of outcomes you care about by the total number of outcomes.

$$\text{probability} = \frac{\text{number of outcomes you care about}}{\text{total number of outcomes}}$$

Sometimes it is difficult to count either or both the numerator and denominator of this fraction. This is where the counting techniques you have learned in the previous sections will come in handy.

19. Exeter 86:2.

20. Exeter 86:8.

21. Say that you have four children, and that each is equally likely a boy or girl. Is it more likely that all will be one sex, that three will be one sex and one the other, or that you'll have two of each sex? It is worthwhile to write out all possibilities—how many are there? What's the exact probability of each scenario?

22. Exeter 88:11.

23. Exeter 89:8.

24. (2010 AMC 10A, problem 18). Bernardo randomly picks 3 distinct numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and arranges them in descending order to form a 3-digit number. Silvia randomly picks 3 distinct numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ and also arranges them in descending order to form a 3-digit number. What is the probability that Bernardo's number is larger than Silvia's number?