Math 115A
Homework 1 Comments

I graded 10 of the problems:
Section 1.2: 9, 11, 13, 14, 17
Section 1.3: 2, 10, 18, 19, 22.

Each problem is worth 2 points. A grade of 0 indicates no solution or a substantially wrong solution. A grade of 2 indicates a correct or nearly correct solution. Otherwise the grade given is 1.

If you believe a problem was misgraded, or I made some addition or other error, please write a short note explaining the situation, attach it to your homework, and return it to me (either in person, in my mailbox, or under my office door). I'll take a look and afterwards leave your homework in a box outside my office.

The following are comments and occasionally solutions for the graded problems.

1.2

9. For the corollaries, you assume there two of the item in question and show that they are in fact equal. For example assume there are two additive identitities 0 and $0'$; from VS3 we then have $0 + 0' = 0$ and $0' + 0 = 0'$. By VS1 $0 + 0' = 0' + 0$ and so it follows $0 = 0'$.

   For corollary 2, some people used the notation $-x$ for the additive inverse, even though it is really not defined until we've used corollary 2 to show that the inverse is unique. However I accepted it as just a notation for "one of" the additive inverses.

   For 1.2c, most people used the trick that $a0 = a(0+0) = a0 + a0$ and cancelling by 1.1 we get $a0 = 0$.

11. This was a giveaway. As long as you looked like you went through the axioms and made a reasonable effort I gave full points.

13. This was a very good problem in that it revealed some subtleties about vector spaces. A lot of people assumed the identity element was $(0, 0)$, for example. But all we need is the existence of some element that satisfies V3, and in fact the only such element is $(0, 1)$.

    To prove this is not a vector space, you just have to show a single axiom, or some theorem that holds for all vector spaces (like 1.2) fails. Theorem 1.2a is perhaps the easiest to use, since $0(a_1, a_2) = (0, a_2) \neq (0, 1)$ (the last being the vector space additive identity) when $a_2 \neq 1$. Notice in 1.2a that there are two different 0's involved, one being an element of the field and the other the notation for the vector space additive identity. It's confusing, so be careful.

    Most people guessed that V4 would fail (no doubt since it says so in the back of the textbook!) but weren't able to justify it. If (a, b) has an inverse (c, d), then $(a, b) + (c, d) = (0, 1)$ which implies $a + c = 0$ and $bd = 1$. But the latter is impossible if $b = 0$, so elements of the form $(a, 0)$ don't have inverses. VS8 also fails for most $x \in V$ as you can check.

14. This problem also caused some confusion. For example some people tried to claim V over $\mathbb{R}$ is a subspace of V over $\mathbb{C}$, but a subspace must be over the same field as the original space! You are only taking a subset of the vectors when making a subspace.

    The best answer would be to point out which of the vector space axioms do not depend on scalar multiplication, and say those are unaffected. Then for the others show carefully that each still holds when the scalar is in $\mathbb{R}$ rather than $\mathbb{C}$ (because of the "for all" quantifiers and the fact that a real number times a complex number is a complex number). I was fairly generous in grading this one, but make sure you understand what's going on.

17. Unlike 13, most people had no trouble with this one. VS5 clearly fails as $1(a_1, a_2) = (a_1, 0)$ by definition, but VS5 says it must equal (a_2, a_2), so this is not true for $a_2 \neq 0$. Note that V8 also fails in general; the other axioms all hold.

1.3

2. No one had trouble here, although a few people forgot to compute the trace of each matrix.

10. Again most people had no trouble with this. W_2 fails all three properties of a subspace, but $0 \notin W_2$ is the easiest to show.

18. A lot of people got one direction ($\Longrightarrow$) but had trouble with the other. The key is the use the fact that $0 \in W$ and $1 \in F$, and their properties. For many people it was confusing to me to figure out which part of the proof they were proving. Please be clear about this. The following is what a good proof might look like.

    Proof. A subset $W \subseteq V$ is a subspace iff it satisfies the following properties:

    (a) $0 \in W$.

    (b) For all $x, y \in W$, $x + y \in W$.

    (c) For all $a \in F$ and $x \in W$, $ax \in W$.

    Consider the following properties of a subset $W \subseteq V$:

    (1) $0 \in W$.

    (2) For all $a \in F$ and $x, y \in W$, $ax + y \in W$.

    We want to show that (a,b,c) hold iff (1,2) hold.

    ($\Longrightarrow$) (a) implies (1). Given $a \in F$ and $x, y \in W$, by (c) we have $ax \in W$. Now by (b) we have $ax + y \in W$, so (2) holds.

    ($\Longleftarrow$) (1) implies (a). Since $1 \in F$, (2) implies $1x + y = x + y$ for all $x, y \in W$, so (b) holds. Since $0 \in W$, (2) implies $ax + 0 = ax \in W$ for all $a \in F$, $x \in W$ and so (c) holds.

    □

19. This was surely the hardest problem on the homework. Although I went over it in office hours most people who followed my proof used language that suggested they didn't fully understand the proof, so I didn't give the full 2 points. Most people did get the easy direction. For the hard direction, as I mentioned in recitation arguing the contrapositive seems to be less confusing. Here is what a good proof might look like.

    Proof. ($\Longleftarrow$) If $W_1 \subseteq W_2$ then $W_1 \cup W_2 = W_2$, which is a subspace. Similarly if $W_2 \subseteq W_1$ then $W_1 \cup W_2 = W_1$, which is a subspace.

    ($\Longrightarrow$) We will prove the contrapositive, namely that if $W_1 \not\subseteq W_2$ and $W_2 \not\subseteq W_1$, then $W_1 \cup W_2$ is not a subspace. Now if the former holds there must exist some $x \in W_1 \setminus W_2$ (meaning $x \in W_1$ but $x \notin W_2$) and $y \in W_2 \setminus W_1$. Both $x, y \in W_1 \cup W_2$, so if $W_1 \cup W_2$ were a subspace then $x + y = z \in W_1 \cup W_2$.

    Now if $z \in W_1 \cup W_2$ then it must be the case that either $z \in W_1$ or $z \in W_2$. Say $z \in W_1$. Then since also $x \in W_1$ and thus $-x \in W_1$, we have $z - x = y \in W_1$, a contradiction. Similarly if $z \in W_2$ then it would follow $x = z - y \in W_2$, also a contradiction.

    □

22. Most people got this right. A few forgot to show that the zero function ($f(x) = 0$ for all $x \in F_1$), which is the identity for the vector space of functions, is both even and odd, but I didn't take off for this.