Math 115A
Homework 2 Comments

I graded 10 of the problems:
Section 1.4: 3d, 4b, 5b, 10, 15
Section 1.5: 5, 13b, 20
Section 1.6: 2d, 4, 13

Each problem is worth 2 points. A grade of 0 indicates no solution or a substantially wrong solution. A grade of 2 indicates a correct or nearly correct solution. Otherwise the grade given is 1.

If you believe a problem was misgraded, or I made some addition or other error, please write a short note explaining the situation, attach it to your homework, and return it to me (either in person, in my mailbox, or under my office door). I'll take a look and afterwards leave your homework in a box outside my office.

The following are comments and occasionally solutions for the graded problems.

General Comments
Although the median score was down slightly from the previous problem set (11 versus 12 before), there seemed to be more high scores than before, and overall I felt the quality was higher. I was impressed that peoples' proofs already seem better than the first assignment. The high score this time was 19; the high score of homework 1 was 17.

1.4

3d. I graded this part since it wasn't in the back of the book. If you got it right and showed your work (I'm happy to say almost everyone showed some work) then you got 2 points.

4b. Same as 3d.

5b. Same as 3d.

15. I gave one point for a correct proof and one point for two good examples. If you got most of the proof and just one example, I gave one point overall. Quite a few people did very well on the proof, and there were several good solutions. The easiest was to use problem 13 in the same section. Since $S_1 \cap S_2 \subseteq S_1$ and also $S_1 \cap S_2 \subseteq S_2$, then we have both $\mathrm{span}(S_1 \cap S_2) \subseteq \mathrm{span}(S_1)$ and $\mathrm{span}(S_1 \cap S_2) \subseteq \mathrm{span}(S_2)$, so $\mathrm{span}(S_1 \cap S_2) \subseteq \mathrm{span}(S_1) \cap \mathrm{span}(S_2)$. You can also use Theorem 1.5, or argue directly from the definition of span.

    Not many people were able to come up with good examples. I'd like to emphasize that it's very important to be able to come up with examples and counterexamples, both to make sure you understand what is going on and to help your intuition when you run into more abstract mathematics. It's a hard skill to develop, but as with anything the best way to get better is to practice.

    For an example of where equality holds, taking $S_1 = S_2$ (say $S_1 = S_2 = \{(0,1)\} \subseteq \mathbb{R}^2$) will certainly work, since then both sides are simply $\mathrm{span}(S_1)$. For inequality it's a little harder. A good idea is to choose S_1 and S_2 so that they have empty intersection but that their spans do not. So for example $S_1 = \{(0,1)\}$ and $S_2 = \{(0,2)\}$ (in $\mathbb{R}^2$; for examples always make it clear what vector space you're working in!) would work as you can check.

1.5

5. I regretted a little picking this one to grade since it's hard to say exactly what a good answer is. I ended up being quite lenient with grading. Given $a_n x^n + \cdots + a_0 = 0$, we want to show $a_i = 0$ for all i. Now first of all what is 0 in the vector space $P_n(F)$? It's the zero polynomial. We know that two polynomials are equal iff all of their coefficients are equal, and $0 = 0x^n + \cdots + 0$, so it follows all $a_i = 0$.

13b. This problem was quite difficult and no one got it perfectly correct, although I'm happy that a few people came very close, and many people got the easy direction at least. I went over the solution to part (a) in office hours, so I just graded part (b), which is a more difficult variation on part (a).

    First of all what does "a field of characteristic not equal to two" mean? I looked at the sections referenced by the index, but nowhere does the text seem to define characterstic of a field. You'll learn this in Math 110 or 117, but basically characteristic two means that in the field $2 = 0$, where

$2 = 1 + 1$ and 0 and 1 are the additive and multiplicative identities, respectively. In all the fields we deal with in this class the characterstic is 0, which means you can add 1 to itself as many times as you like and you'll never get 0. However there exist fields of "positive" characterstic such as 2; an example is the finite field with two elements 0 and 1, and where addition and multiplication are defined modulo 2. These fields have important applications, for example in coding theory. The problem with them, however, is that you can't divide by 2, and it turns out that will be necessary in both parts of this problem, so that is why they are ruled out.

The first difficulty of this problem is a logical one. Linear independence is phrased as an implication: $\sum a_i x_i = 0 \implies a_i = 0$. On top of that we are trying to show something LI implies something else LI, so the form of what we're trying to prove (just one direction) is like this:
$(A \implies B) \implies (C \implies D)$. This is very confusing! If you're not careful you might end up proving instead something more like $(A \implies B) \implies C$ or $(A \implies B) \implies D$, both of which are not what we want.

If you think carefully about what $(A \implies B) \implies (C \implies D)$ means, though (or manipulate it logically, or try all 16 values of true and false), you'll see that it's logically equivalent to
$[(A \implies B) \wedge C] \implies D$, where $\wedge$ means "and". One way to reason is this: We are trying to show $C \implies D$, given that $A \implies B$. Now to show that $C \implies D$, we need to show that D holds whenever C holds. However we also know that $A \implies B$ holds, so this is the same as showing if both $A \implies B$ and C hold, then D holds.

Now finally to the problem. For iff proofs, the strategy to use is to note that one direction is almost always easier than the other, and to find that easy direction and do it first. Not only do you get partial credit that way (I gave one point for each direction), but you can often understand what's going on better and then be in better shape to tackle the harder part. In this case the harder part involves some further technical difficulties. Here is the full proof.

Proof. ($\implies$) We are given u, v, w are LI, which means $a_1 u + a_2 v + a_3 w = 0 \implies a_1 = a_2 = a_3 = 0$. We want to show $u + v, u + w, v + w$ are LI, which means that
$b_1(u + v) + b_2(u + w) + b_3(v + w) = 0 \implies b_1 = b_2 = b_3 = 0$. Rearranging terms we have
$(b_1 + b_2)u + (b_1 + b_3)v + (b_2 + b_3)w = 0$. Since u, v, w are LI by hypothesis, we know that
$b_1 + b_2 = b_1 + b_3 = b_2 + b_3 = 0$. We can easily solve this system of 3 equations in 3 unknowns to find that $b_1 = b_2 = b_3 = 0$.

Note that even in this part the field characteristic not being 2 comes into play. When you solve the system you'll get something like $2b_3 = 0$, and you'd conclude b_3 must be 0, but you can't do that if $2 = 0$. In fact in a field of characteristic 2, $b_1 = b_2 = b_3 = 1$ is also a solution as you can easily check.

($\impliedby$) Now we are given $u + v, u + w, v + w$ are LI, which means that
$a_1(u + v) + a_2(u + w) + a_3(v + w) = 0 \implies a_1 = a_2 = a_3 = 0$. Again we can rearrange terms so that
$(a_1 + a_2)u + (a_1 + a_3)v + (a_2 + a_3)w = 0 \implies a_1 = a_2 = a_3 = 0$. We want to prove that u, v, w are LI, which means $b_1 u + b_2 v + b_3 w = 0 \implies b_1 = b_2 = b_3 = 0$. The problem here is that to use our hypothesis, we need to find, given b_1, b_2, b_3, a triplet of numbers a_1, a_2, a_3 such that $a_1 + a_2 = b_1$, $a_1 + a_3 = b_2$, and $a_2 + a_3 = b_3$. If such a triplet exists, then by our hypothesis we know that $a_1 = a_2 = a_3 = 0$, which will immediately imply that $b_1 = b_2 = b_3 = 0$, what we want.

How do we find such a triplet? By linear algebra of course! In matrix form we have

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Call the matrix on the left A. Then to solve this system for a_1, a_2, a_3 we can either row-reduce, or multiply both sides of the equation by A^{-1}. I won't do the computation here, but the result is that a solution exists and it is $a_1 = \frac{b_1}{2} + \frac{b_2}{2} - \frac{b_3}{2}$, $a_2 = \frac{b_1}{2} - \frac{b_2}{2} + \frac{b_3}{2}$ and $a_3 = \frac{-b_1}{2} + \frac{b_2}{2} + \frac{b_3}{2}$. Again we see where characteristic not two is important.

□

If you know anything about change of basis this may all look familiar. What we are proving is that $u + v, u + w, v + w$ is also a basis for the space spanned by u, v, w, and the reason this is true is that the change of basis matrix (which also happens to be A) is invertible, as you can tell just by calculating its determinant (which happens to be -2; so you see again the matrix would not be invertible in a field of characteristic 2). That gives the quickest proof of this problem, so I guess the point of doing it the hard way now is to appreciate how much easier things become when we've built up some more theory.

20. To do this problem we both have to understand just what the vector space is, and also some properties of the exponential function. First of all what is the 0 element of the vector space $F(\mathbb{R}, \mathbb{R})$? It is a function, call it f_0, such that $f_0(x) = 0$ for all $x \in \mathbb{R}$. So to say that f and g are LI means that $ae^{rt} + be^{st} = f_0$ implies that $a = b = 0$.

    Now if $ae^{rt} + be^{st} = f_0$, this implies $ae^{rt} = -be^{st}$. Say $b \neq 0$. Then it follows $-be^{st}$ is never 0 for any $t \in \mathbb{R}$, so we can divide both sides by $-be^{st}$. Note that this is not a vector space operation–we are treating this as an equality of functions of real numbers now. We get that $\frac{-a}{b}e^{(r-s)t} = 1$ for all $t \in \mathbb{R}$. However since by hypothesis $r \neq s$, this is not true. So it must be the case that $b = 0$. Now we are reduced to $ae^{rt} = f_0$, and the only way for this to be true is for $a = 0$.

1.6

2d. Grading was the same as problem 3d of section 1.4. This is a basis. Some people checked both linear independence and that the set spans, but just one is enough because you know a basis of $\mathbb{R}^3$ has 3 elements.

4. This has a very simple solution: The dimension of $P_3(\mathbb{R})$ is 4, and there are only three vectors (polynomials) in this set. So they cannot possibly generate (span) $P_3(\mathbb{R})$. Some people thought this problem was asking if the polynomials are linearly independent, which is true, but not what was asked for.... Perhaps they thought the dimension of $P_3(\mathbb{R})$ was 3, in which case proving linear independence would have been sufficient to prove that the vectors span the space. Some people actually worked through trying to generate any polynomial in $P_3(\mathbb{R})$ from the three, and I gave full credit for this even though it is certainly not the best solution.

11. I was originally going to grade this, but decided it was too similar to problem 13 of section 1.5, and so graded problem 13 of this section instead. Note that you can use the same method of 1.5 number 13 to solve this–prove that the vectors are linearly independent, and then it follows they are a basis since there are the same number of them as the original basis $\{u, v\}$. Another way to solve it would be to show that the sets span the same space as $\{u, v\}$.

13. Most people had no trouble with this one. You row-reduce the matrix and find the basis of the null space is $\{(1, 1, 1)\}$.