Math 132
Homework 3 Comments

I graded 4 of the problems:
Section 1.5: 2
Section 2.6: 1a, 2
Section 2.7: 1d

Each problem is worth 3 points. A grade of 0 indicates no solution or a substantially wrong solution. A grade of 3 indicates a correct or nearly correct solution. Otherwise the grade given is 1 or 2 depending upon how much work was put in and how close the solution is to being correct. How neat and clear your solution is also affects the grade I give.

If you believe a problem was misgraded, or I made some addition or other error, please write a short note explaining the situation, attach it to your homework, and return it to me (either in person, in my mailbox, or under my office door). I'll take a look and return the homework in the next section.

The following are comments and occasionally solutions for the graded problems.

General Comments
The maximum number of points was 12. The high score was 12, and the median was 7.
Yet again I graded two algebraic problems and two graphing problems. Of each I picked what I thought was a harder problem (2.6 1a and 2.7 1d) and an easier problem (1.5 2 and 2.6 2). This was a harder problem set than last time and as expected the scores were lower, although some people still did very well.

2.5 2. This is a simple problem if you understand what harmonic conjugates are, but many people had trouble with it. First of all if v is a harmonic conjugate of u, then both u and v are harmonic and $u + iv$ is analytic. To show $-u$ is a harmonic conjugate of v, we need to show both v and $-u$ are harmonic, and that $v - iu$ is analytic. It is immediate that v is harmonic, and for $-u$ we have $\frac{\partial^2(-u)}{\partial x^2} + \frac{\partial^2(-u)}{\partial y^2} = -(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) = 0$ since u is harmonic. You needed to at least mention this to get full credit.

To show $v - iu$ is analytic the simplest method is to note that $v - iu = (-i)(u + iv)$. Now the constant function $-i$ is analytic as is $u + iv$ by hypothesis, and the product of two analytic functions is analytic, so we're done. You can also show that the Cauchy-Riemann equations for $v - iu$ hold given that they hold for $u + iv$.

2.6 1. I just graded part (a). I personally didn't find the hint to use polar coordinates very enlightening, and I stuck with rectangular coordinates as did those people who got it right. It took me a while playing with it to figure out what was going on, but once I did it made sense, and this is a good problem to understand.

First note that if $f(z) = 1/z$ then $f'(z) = -1/z^2$ at all points except $z = 0$, so $f(z)$ is conformal everywhere except at $z = 0$. You needed to say this to get full credit.

Now writing $z = x + iy$, we have $f(z) = 1/(x + iy) = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$. Now if $u(x, y)$ is constant we can let it equal any constant, so anticipating making things work out nicely we let $\frac{x}{x^2+y^2} = \frac{1}{2c}$ where $c \neq 0$. We then get $x^2 - 2cx + y^2 = 0$, so completing the square we have $x^2 - 2cx + c^2 + y^2 = (x - c)^2 + y^2 = c^2$, which we recognize as the equation of a circle centered at $(c, 0)$ with radius c (so it touches the origin). Such circles exist for all $c \neq 0$. There is also a degenerate case for $\frac{x}{x^2+y^2} = 0$. The level curves for $v(x, y)$ are similar (try working it out if you haven't already), except the circles are now centered at $(0, c)$ with radius c. The diagram on page 67 shows roughly what is going on, save the circles should be shifted so that they are all interesecting at the origin.

2.6 2. This was any easy problem I hoped everyone would get right, and most people did. We have $\operatorname{Log} z = \log |z| + i \operatorname{Arg} z$, so $|z| = e^c$ is a circle of radius e^c, and $\operatorname{Arg} z = c$ is a ray from the origin of angle $\operatorname{Arg} z$. The level curves look like those in the first diagrams on page 63.

The problem didn't ask about where $\operatorname{Log} z$ is conformal, but some people claimed anyway that it is analytic everywhere. This is wrong, however–it's not even defined at $z = 0$ and the negative real axis. We need to make some kind of branch cut like this or the function cannot even be continuous, much less analytic. It is however analytic on the rest of $\mathbb{C}$.

2.7 1. I just graded part (d). Part (a) was done by Prof. Effros on the course webpage, (b) was done in section, and (c) was straightforward, so this was the logical choice. It was fairly hard, too, although once you see the answer you wish you'd guessed it and saved all the work. Some people did just write down the answer without showing any work, but I only gave one point for that.

The standard way to solve this problem is to send $(-2, i, 2) \mapsto (0, 1, \infty)$ via a linear fractional transformation (LFT) f, send $(1-2i, 0, 1+2i) \mapsto (0, 1, \infty)$ via an LFT g, and then the desired map will be $g^{-1} \circ f$. The problem with this is that the algebra gets rather messy, and it's easy to make a mistake which will make things messier still. However most people who solved the problem (or got close) did it this way, and I only took off a point if there was a mistake somewhere.

For a simpler solution, we have to be a little clever. The key is that since we know $i \mapsto 0$ we can use that to simplify things. If we call our desired LFT $f(z) = \frac{az+b}{cz+d}$, we know that the numerator must be $a(z-i)$. Now if $a = 0$ then $f(z) = 0$, which we know is not true, so let's divide top and bottom by a and rename the constants in the denominator. So we have $f(z) = \frac{z-i}{cz+d}$.

We now plug in $z = \pm 2$, hoping to get two linear equations in two unknowns, which we can solve. Sure enough we get (you can fill in the details) $-2c + d = -i$ and $2c + d = -i$. Adding these equations together we get $d = -i$ and then we must have $c = 0$. So the solution is $f(z) = \frac{z-i}{i} = iz + 1$.