Math 132
Homework 5 Comments

I graded 4 of the problems:
Section 4.4: 3
Section 4.5: 4
Section 5.1: 3
Section 5.2: 8

Each problem is worth 3 points. A grade of 0 indicates no solution or a substantially wrong solution. A grade of 3 indicates a correct or nearly correct solution. Otherwise the grade given is 1 or 2 depending upon how much work was put in and how close the solution is to being correct. How neat and clear your solution is also affects the grade I give.

If you believe a problem was misgraded, or I made some addition or other error, please write a short note explaining the situation, attach it to your homework, and return it to me (either in person, in my mailbox, or under my office door). I'll take a look and return the homework in the next section.

The following are comments and occasionally solutions for the graded problems.

General Comments
The maximum number of points was 12. The high score was 10, the median was 3 and the mean was 4. I didn't grade anything from section 4.2 since they were either straightforward or we covered them in section. I picked three harder problems and one easy problem from the remaining four sections. The hard problems gave most everyone trouble and as a result this was by far the lowest-scoring homework so far.

4.4 3. No one had a perfect solution for this problem but several people came fairly close and got full credit. The most common error was to assume u was analytic, which is not true–it's a function from $\mathbb{R}^2$ to R; if it were analytic its harmonic conjugate would be 0, which is a very special case! I gave only one point for this. Some people magically conjured up an analytic function f and then claimed that implied u satisfies the mean value property from that, without explaining the relationship between f and u. It thought that seemed a little closer to the right answer and gave two points for that answer, although it still misses the key points.

Here is the solution, then. Since the closed disk $|z - z_0| \leq \rho$ is contained in D and D is an open set, there exists some $\rho' > \rho$ such that the open disk $|z - z_0| < \rho'$ is contained in D. Call this open disk D'. Now by using either the theorem in section 2.5 or section 3.3, u has a harmonic conjugate v on D'. Note that we can't use the original domain D which might have some strange shape. By the definition of harmonic conjugate, there exists an analytic function f on D' such that $f = u + iv$. Finally let D'' be the open disk $|z - z_0| < \rho$. This is what we'll integrate over. We needed to use the bigger D' to define f however to make sure that f extends properly to $\partial D''$.

We now use Cauchy's integral formula to deduce $f(z_0) = \frac{1}{2\pi i} \int_{\partial D''} \frac{f(z)}{z - z_0} dz$. We parameterize z on $\partial D''$ as $z = z_0 + \rho e^{i\theta}$. Then $dz = i\rho e^{i\theta} d\theta$. This gives

$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + \rho e^{i\theta})}{\rho e^{i\theta}} i\rho e^{i\theta} \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{i\theta}) \, d\theta.$$

Writing both sides in terms of u and v, we get

$$u(z_0) + iv(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + \rho e^{i\theta}) \, d\theta + i \frac{1}{2\pi} \int_0^{2\pi} v(z_0 + \rho e^{i\theta}) \, d\theta.$$

Now we have to note that since u and v are both real-valued, their integrals are also real-valued. Therefore since the real parts of each side must be equal we get the result.

4.5 4. This problem gave everyone problems–no one got more than one point for it. In fact no one even got the easier second part correct. The book gives a hint but there's still work to be done and I didn't give any credit for just copying the hint. Here's how the solution goes.

I assume R is a fixed constant in the problem statement, so it's confusing that Gamelin is varying it in his hint. Let's let r be a variable radius. Also let z_0 be an arbitrary point in $\mathbb{C}$. We are given that there exists some constant M such that $|f(z)/z^n| \leq M$ for $|z| \geq R$. Now let r be large enough so that all points of $|z - z_0| = r$ satisfy also $|z| \geq R$. Consider the value of $|f(z)/z^n|$ on the circle $|z - z_0| = r$. We have $|f(z)/r^n| \leq M$, which implies $f(z) \leq r^n M$. The right hand side is a constant (for the points $|z| = r$), and $f(z)$ is entire, so we can apply the Cauchy Estimates to get

$$\left|f^{n+1}(z_0)\right| \leq \frac{(n+1)!}{r} M$$

Now as $r \to \infty$ it follows $\left|f^{n+1}(z_0)\right| \to 0$ for all z_0, so $f^{n+1}(z) = 0$. Integrating $n+1$ times we find that $f(z)$ is a polynomial of degree at most n.

For the second part, most people tried to use Liouville's Theorem, but it doesn't necessarily apply because we don't know a priori that $f(z)/z^n$ is entire. So we have to argue directly. If $f(z)/z^n$ is bounded on the entire complex plane then it is certainly bounded for all $|z| > 1$, say, so the first part holds and we know $f(z) = \sum_{k=0}^{n} a_i z^i$. This gives us $f(z)/z^n = a_n + \frac{a_{n-1}z^{n-1} + \cdots + a_1 z + a_0}{z^n}$. Now if any of the a_i are nonzero for $0 \leq i \leq n-1$, then $f(z)/z^n$ cannot be bounded at 0, since the fraction will go to infinity as $z \to 0$ (the numerator will remain nonzero as the denominator goes to 0), so it must be the case that all these coefficients are 0. Therefore $f(z)/z^n = a_n$ and it follows $f(z) = a_n z^n$.

5.1 3. This problem was not too hard but you had to approach it the right way, otherwise you run into trouble. Very few people got it right, but I gave generally one point for a reasonable attempt. The key point is to note that $S = \sum_{k=1}^{\infty}$, so $\left|S - \sum_{k=1}^{n} \frac{1}{k^p}\right| = \left|\sum_{k=n+1}^{\infty} \frac{1}{k^p}\right| = \sum_{k=n+1}^{\infty} \frac{1}{k^p}$ since all the terms are positive. We now estimate this using the integral which gives $\sum_{k=n+1}^{\infty} \frac{1}{k^p} \leq \int_n^{\infty} \frac{dx}{x^p} = \left.\frac{x^{1-p}}{1-p}\right|_n^{\infty} = 0 - \frac{n^{1-p}}{1-p} = \frac{1}{(p-1)n^{p-1}}$ since $p > 1$. Since this goes to 0 as $n \to \infty$ this proves that the series converges to S.

5.2 8. After three hard problems this one was intended to be a giveaway, and I was very generous grading it as well. As the hint in the back of the book says, use the M-test. There is really only one reasonable thing to compare the terms to: $\left|\frac{z^k}{k^2}\right| = \frac{|z^k|}{k^2} < \frac{1}{k^2}$ since $|z| < 1$. So let $M_k = \frac{1}{k^2}$, and note that $\sum M_k$ converges. You can just claim this as a known fact—I didn't require a proof of it. Now the M-test applies to show $\sum \frac{z^k}{k^2}$ converges uniformly.