Math 132
Homework 7 Comments

I graded 4 of the problems:
Section 6.1 1b
Section 6.2: 1a
Section 7.1: 1d
Section 7.2: 4

Each problem is worth 3 points. A grade of 0 indicates no solution or a substantially wrong solution. A grade of 3 indicates a correct or nearly correct solution. Otherwise the grade given is 1 or 2 depending upon how much work was put in and how close the solution is to being correct. How neat and clear your solution is also affects the grade I give.

If you believe a problem was misgraded, or I made some addition or other error, please write a short note explaining the situation, attach it to your homework, and return it to me (either in person, in my mailbox, or under my office door). I'll take a look and return the homework in the next section.

The following are comments and occasionally solutions for the graded problems.

General Comments

The maximum number of points was 12. The high score was 12, the median was 8.7 and the mean was 9. Only about 2/3 of the class handed in this last assignment (the statistics only include those assignments graded), but those who handed it in did fairly well. The assignment was not too hard and there were several 12's and 11's.

6.1 1. I graded part (b). Our function is $\frac{z-1}{z+1}$. This has a singularity only at $z=-1$, so we need to find two separate Laurent expansions for the regions $|z|<1$ and $|z|>1$. Some people said things like $|z|<-1$ or $0<|z|<1$ which are both incorrect, but I didn't take off for them. I also didn't worry about exact calculations too much, but I did only give one point if you didn't realize you hand to handle two cases.

For each case there are several ways to proceed. The easiest, which I didn't do but which several students did, is to first realize that $\frac{z-1}{z+2} = 1 - \frac{2}{z+1}$. This just makes the calculations a little simpler. We want to expand this first as a power series that converges for $|z|<1$. This is easy if we notice that $1-\frac{2}{z+1} = 1 - \frac{2}{1-(-z)} = 1-2(1-z+z^2-z^3+\cdots) = -1+\sum_{k=1}^{\infty}(-1)^{k-1}2z^k$.

For $|z|>1$ we have $|1/z|<1$, so we expand as a geometric series in $1/z$. We have $1-\frac{2}{z+1} = 1-\frac{2}{z}\frac{1}{1-(-1/z)} = 1-\frac{2}{z}(1-1/z+1/z^2-1/z^3+\cdots) = 1-2/z+2/z^2-2/z^3+2/z^4-\cdots = 1+\sum_{k=1}^{\infty}(-1)^k 2/z^k$.

6.2 1. I graded part (a). Most people got this right, aided by the solution in the back of the book. Almost everyone used partial fractions, which I guess was emphasized in class. So you try to write $z/(z^2-1)^2 = \frac{z}{(z-1)^2(z+1)^2}$ as $\frac{A}{z-1}+\frac{B}{(z-1)^2}+\frac{C}{z+1}+\frac{D}{(z+1)^2}$, and after griding through some algebra you get $A=C=0$, $B=1/4$ and $D=-1/4$. This gives us the Laurent expansion about $z=1$ which is $\frac{1/4}{(z-1)^2}+f(z)$ where $f(z)=\frac{-1/4}{(z+1)^2}$ and $f(z)$ is analytic in a small enough disc around $z=1$, so we can read off immediately that the pole at $z=1$ is of order 2 (a double pole) and that the principal part is $\frac{1/4}{(z-1)^2}$. Similarly the Laurent expansion about $z=-1$ is $\frac{-1/4}{(z+1)^2}+g(z)$ where where $g(z)=\frac{1/4}{(z-1)^2}$ and $g(z)$ is analytic in a small enough disc around $z=-1$. Therefore the pole at $z=-1$ is also of order two and the principal part is $\frac{-1/4}{(z+1)^2}$.

Here's another way to approach the problem that I like better; it is similar to the method used in rules 1 and 2 for calculating residues in section 7.1. First ascertain that both ± 1 are double poles by proving that they are double zeros of $(z^2-1)^2/z$ (this relies on the theorem in the middle of page 173). Now we know the Laurent series around $z=1$ must be of the form $z/(z^2-1)^2 = a_{-2}(z-1)^{-2}+a_{-1}(z-1)^{-1}+a_0+\cdots$. To calculate the principal part we just need to calculate a_{-2} and a_{-1}. How do we do this? Well we can first convert this to a Taylor series by multiplying both sides by $(z-1)^2$. We get $z/(z+1)^2 = a_{-2}+a_{-1}(z-1)+a_0(z-1)^2+\cdots$. Now we know how to calculate the coefficients of a Taylor series. To find a_{-2} we just plug in $z=1$ to get $a_{-2}=1/4$. To find a_{-1} we differentiate both sides and then plug in $z=1$ which gives $a_{-1}=0$. So we get the same principal part as we did with the other method. As an exercise try this method out for $z=-1$.

7.1 1. I graded part (d). The answer wasn't in the back, but most people got this right. Perhaps the simplest method was to just note that the Taylor series of $\sin z = z - z^3/3 + \cdots$, so $\frac{\sin z}{z^2} = 1/z - z/3 + \cdots$, and we can read off the residue immediately as 1. You can also try out the rules of section 7.1, although only rules 1 and 2 apply. Note that although it might look like $\frac{\sin z}{z^2}$ has a double pole at 0, it only has a single pole, because $\sin z$ has a single zero at 0. So rule 1 is the best to use, and we have the residue equal to $\lim_{z\to 0} \frac{\sin z}{z} = 1$ as you can evaluate using l'Hôpital's rule. Many people claimed a double pole and used rule 2, which happens to work fine (despite the fact that there's only a single pole) although it's overkill. I gave full credit as long as you included enough detail.

7.2 4. Since the degree of the denominator (4) is at least two more then the degree of the numerator (0), we can calculate the integral as $\int_{-\infty}^{\infty} \frac{1}{z^4+1} = 2\pi i \sum_j \text{Res}[\frac{1}{z^4+1}, z_j]$, where the sum is over the poles in the upper half plane. Some people went over the justification of this, which is great to do, but I didn't require it for full credit.

From our previous assignment we know the roots of $z^4 + 1$ are $e^{\pm\pi i/4}$ and $e^{\pm 3\pi i/4}$; only $e^{\pi i/4}$ and $e^{3\pi i/4}$ are in the upper half plane. Using rule 4 we calculate the residue $\text{Res}[\frac{1}{z^4+1}, e^{\pi i/4}] = \frac{1}{4z^3}\big|_{e^{\pi i/4}} = \frac{1}{4}e^{-3\pi i/4} = \frac{1}{4}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$. Similarly we get $\text{Res}[\frac{1}{z^4+1}, e^{3\pi i/4}] = \frac{1}{4}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$. Thus we have

$$\int_{-\infty}^{\infty} \frac{1}{z^4+1} = 2\pi i \left[\frac{1}{4}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) + \frac{1}{4}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)\right] = \frac{\pi}{\sqrt{2}}.$$