Midterm 1 Solutions

1. **(10 points)** Find the derivative of the function

$$f(x) = (\sin x)^{\sqrt{\ln x}}.$$

Solution. Let $y = f(x)$. Then $\ln y = \sqrt{\ln x}\ln(\sin x)$ and

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= \frac{\ln(\sin x)}{2x\sqrt{\ln x}} + \sqrt{\ln x}\cot x \\ \frac{dy}{dx} &= \left((\sin x)^{\sqrt{\ln x}}\right)\left(\frac{\ln(\sin x)}{2x\sqrt{\ln x}} + \sqrt{\ln x}\cot x\right)\end{aligned}$$

2. **(15 points)** How large should n be to guarantee that the midpoint rule error approximation to

$$\int_0^3 \sqrt{x+1}\, dx$$

is accurate to within 0.001?

Solution. Let $f(x) = \sqrt{x+1}$. Then $f'(x) = \frac{1}{2}(x+1)^{-1/2}$ and $f''(x) = -\frac{1}{4}(x+1)^{-3/2}$. We need to bound $|f''(x)|$ as tightly as possible. We have

$$\begin{gathered} 0 \le x \le 3 \\ 1 \le x+1 \le 4 \\ 1 \le (x+1)^{3/2} \le 8 \\ \frac{1}{8} \le (x+1)^{-3/2} \le 1 \\ -\frac{1}{4} \le -\frac{1}{4}(x+1)^{-3/2} \le -\frac{1}{32}. \end{gathered}$$

Since $-1/32 \le 1/4$ our best bound is $|f''(x)| \le 1/4$. We now plug into the formula for midpoint error and get

$$\begin{aligned} 0.001 > |E_M| &\ge \frac{(1/4)3^3}{24n^2} \\ n^2 &> \frac{9000}{32} = \frac{9 \cdot 125}{4} \\ n &> \frac{45\sqrt{5}}{2}. \end{aligned}$$

This answer is sufficent for full credit. Using a calculator one can then show $n \ge 51$ suffices.

3. **(15 points)** Find

$$\int \cos^5 x \, dx.$$

Solution.

$$\begin{aligned}
\int \cos^5 x \, dx &= \int (1 - \sin^2 x)^2 \cos x \, dx \\
&= \int (1 - u^2)^2 \, du \\
&= \int (1 - 2u^2 + u^4) \, du \\
&= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C \\
&= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C
\end{aligned}$$

where we have used the substitution $u = \sin x$, $du = \cos x \, dx$.

4. **(20 points)**

(a) **(10 points)** For what real values $a > 0$ is

$$\lim_{x\to\infty} \frac{x^a \ln x}{x^{10}}$$

equal to 0? Justify your answer.

Solution. We have

$$\lim_{x\to\infty} \frac{x^a \ln x}{x^{10}} = \lim_{x\to\infty} \frac{\ln x}{x^{10-a}} \overset{H}{=} \lim_{x\to\infty} \frac{1/x}{(10-a)x^{10-a-1}} = \lim_{x\to\infty} \frac{1}{(10-a)x^{10-a}}$$

by l'Hôpital's Rule, which in the limit is 0 if and only if $10 - a > 0$; in other words when $a < 10$. Note that $a \le 0$ also results in a limiting value of 0.

(b) **(10 points)** For what real values $b \in (0, 2]$ is

$$\lim_{x\to\infty} \frac{(\ln x)^b}{x}$$

equal to 0? Justify your answer.

Solution. Again use l'Hôpital's Rule:

$$\lim_{x\to\infty} \frac{(\ln x)^b}{x} \overset{H}{=} \lim_{x\to\infty} \frac{b(\ln x)^{b-1}}{x}.$$

If $b \in (0, 1]$, then this limit is 0. Otherwise use use l'Hôpital's Rule again:

$$\lim_{x\to\infty} \frac{b(\ln x)^{b-1}}{x} \overset{H}{=} \lim_{x\to\infty} \frac{b(b-1)(\ln x)^{b-2}}{x}$$

and now for all $b \in (0, 2]$ it is clear that this limit is 0.

Note that you can repeat this argument using induction to show that $\lim_{x\to\infty} \frac{(\ln x)^b}{x} = 0$ for all $b > 0$.

Another way to solve this problem is to note that for for any function $f(x)$ and for any $b > 0$, $\lim_{x\to\infty} f(x) = 0$ if and only if $\lim_{x\to\infty} [f(x)]^{1/b} = 0$. Applying this to our problem, we have

$$\lim_{x\to\infty} \frac{(\ln x)^b}{x} = \lim_{x\to\infty} \frac{\ln x}{x^{1/b}}$$

and now one application of l'Hôpital's Rule suffices to show the limit is 0 for all $b > 0$.

5. **(20 points)** Find

$$\int_1^4 \sqrt{x}e^{\sqrt{x}}\, dx.$$

Solution. Let $t = \sqrt{x}$; then $dt = 1/(2\sqrt{x})\, dx$. It follows $\sqrt{x}\, dx = 2x\, dt = 2t^2\, dt$. Therefore

$$\int_1^4 \sqrt{x}e^{\sqrt{x}}\, dx = 2\int_1^2 t^2 e^t\, dt.$$

Now integrate by parts twice. The first time we have $u = t^2$, $du = 2t\, dt$, $dv = e^t\, dt$, $v = e^t$ and our integral is

$$2(t^2e^t)\Big]_1^2 - 4\int_1^2 te^t\, dt = 8e^2 - 2e - 4\int_1^2 te^t\, dt.$$

The second time we have $u = t$, $du = dt$, $dv = e^t\, dt$, $v = e^t$ and

$$\int_1^2 te^t\, dt = te^t\Big]_1^2 - \int_1^2 e^t\, dt = 2e^2 - e - e^2 + e = e^2.$$

Plugging into the equation above gives

$$\int_1^4 \sqrt{x}e^{\sqrt{x}}\, dx = 4e^2 - 2e.$$

Note that the indefinite form of this integral is problem 63 of Section 8.5 of the textbook. It is also very similar to problem 34 of Section 8.1 which was assigned for homework.

6. **(20 points)** Find

$$\int \frac{x+2}{x^4-16}\, dx.$$

Solution. Factoring the integrand gives

$$\frac{x+2}{x^4-16} = \frac{x+2}{(x+2)(x-2)(x^2+4)} = \frac{1}{(x-2)(x^2+4)}.$$

Now let

$$\begin{aligned} \frac{1}{(x-2)(x^2+4)} &= \frac{A}{x-2} + \frac{Bx+C}{x^2+4} \\ 1 &= A(x^2+4) + (Bx+C)(x-2) \end{aligned}$$

and plugging in $x - 2$ give $A = 1/8$. Now plug in $1/8$ for A and solve for B and C:

$$1 = (1/8 + B)x^2 + (C - 2B)x + (1/2 - 2C)$$

which gives $B = -1/8$ and $C = -1/4$. Therefore

$$\begin{aligned} \int \frac{x+2}{x^4-16}\, dx &= \frac{1}{8}\int \frac{1}{x-2}\, dx - \frac{1}{8}\int \frac{x}{x^2+4}\, dx - \frac{1}{4}\int \frac{1}{x^2+4}\, dx \\ &= \frac{1}{8}\ln|x-2| - \frac{1}{16}\ln(x^2+4) - \frac{1}{8}\tan^{-1}(x/2) + K. \end{aligned}$$