Math 32AH
Homework 1 Solutions

I graded 4 of the problems:
Page 26: 14, 22;
Page 34: 10;
Page 39: 18.

Each problem is worth 3 points. A grade of 0 indicates no solution or a substantially wrong solution. A grade of 3 indicates a correct or nearly correct solution. Otherwise the grade given is 1 or 2 depending upon how much work was put in and how close the solution is to being correct. How neat and clear your solution is also affects the grade I give.

If you believe a problem was misgraded, or I made some addition or other error, please write a short note explaining the situation, attach it to your homework, and return it to me (either in person, in my mailbox, or under my office door). I'll take a look and return the homework in the next section.

The following are solutions to the homework problems and additional comments for the problems I graded. Note that solutions are often brief; if you need more detail please ask in section or office hours.

General Comments
The maximum number of points was 12. The high score was 12, the mean was exactly 6 and the mean was 5. Although these scores don't seem especially high I was overall pretty happy with the solutions to problems I graded. It appears almost everyone put in a good effort. As I promised I graded some of the harder problems. I didn't see any problems that were especially interesting to grade in the first section, so I graded two problems in the second section instead.
I would like to see more explanation in solutions about just what you're doing, what you're trying to show and how you're going about showing it. Writing using lots of words (not just symbols) and full sentences is best. Although much of the math you've learned to this point may have seemed like mostly computation, as it gets more advanced it is much more about proofs and communication of ideas. This year-long sequence is the perfect chance for you to start learning how to really communicate mathematics to others. Study closely how the book presents math, as well as your professors. I'll be happy to help out as well.

Page 15

2a. We did this in section, but see me if you missed section and had trouble drawing this.

3. Solution is in the back of the book.

4. (a) $(5/2, -7/2, 0) - (-1/2, -3/2, 5) = (3, -2, -5)$ and $(-7/2, 13/2, 5) - (-1/2, -3/2, 5) = (-3, 8, 0)$ and as these are not scalar multiples of one another the points are not collinear.
(b) $(-1, 1, 1) - (2, -2, -3) = (-3, 3, 4)$ and $(5, -5, -7) - (2, -2, -3) = (3, -3, -4)$ which equals $-(-3, 3, 4)$ and so the points are collinear.

6. (a) $v + w = (4, 1, -1)$
(b) $v - w = (-2, 3, -3)$
(c) $w - v = (2, -3, 3)$
(d) $-2v + 3w = (7, -7, 7)$
(e) $\frac{1}{3}v - \frac{4}{3}w = (\frac{-11}{3}, 2, -2)$
(f) $|v| = \sqrt{1 + 4 + 4} = 3$
(g) $|w| = \sqrt{9 + 1 + 1} = \sqrt{11}$
(h) $v/|v| = \frac{1}{3}(1, 2, -2)$ and $w/|w| = \frac{1}{\sqrt{11}}(3, -1, 1)$.

8a. $B - A = (-2, 0, 4, 2)$; its length is $\sqrt{4 + 14 + 4} = \sqrt{22}$.

10b. By definition of vector addition $\mathbf{v} + (-\mathbf{v}) = (a, b, c) + (-a, -b, -c) = (a - a, b - b, c - c) = (0, 0, 0) = \mathbf{0}$.

13b. Solution is in the back of the book.

14. Completing the squares, we have
$x^2 - 3x + y^2 + 2y + z^2 - 4z = (x^2 - 3x + \frac{9}{4}) + (y^2 + 2y + 1) + (z^2 - 4z + 4) - (\frac{9}{4} + 1 + 4)$, which means the equation for the sphere is $(x - \frac{3}{2})^2 + (y + 1)^2 + (z - 2)^2 = 9$. The center is $(3/2, -1, 2)$ and the radius is 3.

15. If $\mathbf{v} = \mathbf{0}$ then $\mathbf{u}$ can be any unit vector; otherwise let $\mathbf{u} = \mathbf{v}/|\mathbf{v}|$.

19. If $\mathbf{v} = (r_1, \ldots, r_n)$ then $\mathbf{v} = \sum_{i=1}^n r_i \mathbf{e_i}$. Conversely $\sum_{i=1}^n r_i \mathbf{e_i} = (r_1, \ldots, r_n)$, so the representation is unique.

24. Let V be the set of all differentiable real valued functions on an interval $[a, b]$. We want to show that V is a real vector space, which means showing it satisfies properties (1) to (9) of Theorem 1.11. (Note however that property (9) is superfluous as it follows directly from property (6); see if you can prove this.)

    Each of the properties is straightforward to check. Addition of functions is commutative and associative, and the zero function $\mathbf{0}(x) = 0$ is differentiable and can act as the additive identity. Given $f \in V$ we have $-f \in V$ and $f + (-f) = \mathbf{0}$. For any $f \in V$ and real number $r \in R$ it is clear that $rf \in V$ since it is also real valued and differentiable on $[a, b]$. The distributive laws and properties (7), (8) and (9) are also easy to check.

Page 26

2b. $\theta = \cos^{-1} \frac{4-4}{\sqrt{1/49+9}\sqrt{28^2+16/9}} = \cos^{-1} 0 = \pi/2.$

4b. $\theta = \cos^{-1} \frac{-3+3}{\sqrt{9/4+1}\sqrt{4+64+9}} = \cos^{-1} 0 = \pi/2.$

5. Solution is in the back of the book.

8b. The direction is $\frac{(1,-2,2)}{\sqrt{1+4+4}} = (1/3, -2/3, 2/3)$ and the direction angles are $\cos^{-1} 1/3$, $\cos^{-1} -2/3$, and $\cos^{-1} 2/3$.

9. (a) Letting $\mathbf{v} = (v_1, \ldots, v_n)$, we have $\cos \alpha_i = v_i/|\mathbf{v}|$. It follows $\cos^2 \alpha_1 + \cdots + \cos^2 \alpha_n = (v_1/|\mathbf{v}|)^2 + \cdots + (v_i/|\mathbf{v}|)^2 = \frac{v_1^2+\cdots+v_n^2}{|\mathbf{v}|^2} = 1.$
   (b) Let α be the angle $\mathbf{v}$ makes with the positive x-axis, and let β be the angle $\mathbf{v}$ makes with the positive y-axis (by definition $0 \leq \alpha, \beta \leq \pi$). Then $\beta = \pi/2 - \alpha$ if $\mathbf{v}$ is in the first quadrant, $\beta = \alpha - \pi/2$ if $\mathbf{v}$ is in the second quadrant, $\beta = 3\pi/2 - \alpha$ if $\mathbf{v}$ is in the third quadrant, and $\beta = \alpha + \pi/2$ if $\mathbf{v}$ is in the fourth quadrant. Since $\cos(-\theta) = \cos(\theta)$ for any angle θ, and since $\cos(\alpha + k\pi/2) = \pm \sin \alpha$ for k any odd integer, the result follows.

14. We want their dot product to be zero; in other words $-12/5 - 3/5 + y = 0$; therefore $y = 3$. For the parallel case, one vector needs to be a scalar multiple of the other. Since $-2/\frac{6}{5} = 1/(-\frac{3}{5}) = -\frac{5}{3}$, choosing $y = -\frac{5}{3}$ will make the two vectors parallel.

    Comments: Most people got the first part correct; more people had trouble with the parallel case. A lot of people tried to use the angle to solve these, which is more work than you need since these are special cases that can be solved without cosine. For the parallel case, some people forgot that the cosine could be -1 (which it is in this case) as well as $+1$. I graded this problem on the generous side, giving 2 points out of 3 if you got just one of the two parts right.

16. The component is $(-9 + 6 - 4)/\sqrt{9 + 36 + 4} = -7/7 = -1$.

18. We have $|x - y|^2 = (x - y) \cdot (x - y) = x \cdot x - 2x \cdot y + y \cdot y = |x|^2 + |y|^2 - 2|x||y| \cos \theta$, the last step using Definition 2.7. Note we used bilinearity of the dot product (Theorem 2.4(2)) as well.

20. We have
    $|x + y|^2 - |x - y|^2 = (x+y)\cdot(x+y) - (x-y)\cdot(x-y) = (x\cdot x + 2x\cdot y + y\cdot y) - (x\cdot x - 2x\cdot y + y\cdot y) = 4x\cdot y.$
    The polarization identity is interesting because it shows that you can express the dot product in terms of the norm (length) of vectors. Since norm can be expressed in terms of dot product as well (Corollary 2.5: $|x| = \sqrt{x \cdot x}$) this means the two are in some sense the same—once you have one you get the other for free.

22. Let's fix the origin to be at one vertex of the triangle, so that two of the sides are the vectors x and y, and the third side is the vector $y - x$. Then the midpoints of x and y are simply $x/2$ and $y/2$, and the vector connecting them is $y/2 - x/2$. Since $y/2 - x/2 = \frac{1}{2}(y - x)$, it is parallel to $y - x$ by definition. Furthermore $|y/2 - x/2| = \frac{1}{2}|y - x|$.

Comments: Overall I was quite impressed with the solutions to this problem. The same problem also appeared when I was a TA for 32A last year, and the quality of solutions of this class was much higher than what I saw last year. Some people put all three vertices in an arbitrary position which makes the calculation more complicated than my solution above which places the origin (which can be placed anywhere) at one of the vertices. This is a good lesson to learn now because you'll see again and again in mathematics that a judicious placement of the coordinate axes (or more generally, basis vectors in a vector space, which don't always need to be the standard basis) will save you an enormous amount of work! Note also that choosing the side to concentrate on to be the one whose vertices are both not the origin also makes the computation a lot easier.

Some people were not clear about what they were solving–you need to specifically say what it is that shows the vectors are parallel, and why one is half the length of the other. I took off a point if you were unclear about this, regardless of whether or not the answers could be easily deduced from what you wrote down.

Page 34

2a. $\mathbf{x} = (1, -2) + t(\frac{1}{\sqrt{5}} - 1, -\frac{2}{\sqrt{5}} + 2)$. This is just one possible solution.

4b. $\mathbf{x} = (1, 2, -1) + t(\frac{1}{\sqrt{7}} - 1, \frac{2}{\sqrt{7}} - 2, \frac{\sqrt{2}}{\sqrt{7}} + 1)$. This is just one possible solution.

6. $\mathbf{x} = (1, 4, -2) + t(2, 0, 7)$; $\frac{x-1}{2} = \frac{z+2}{7}$, $y = 4$; $x = 1 + 2t$, $y = 4$, $z = -2 + 7t$. Again this is just one possible solution.

10. If the two lines intersect then $(2, -3, 1) + t(1, 2, -3) = (1, 2, 4) + s(-2, 3, 6)$ for some s, t. This gives three equations in two unknowns:

$$\begin{aligned} 2s + t &= -1 \\ -3s + 2t &= 5 \\ -6s - 3t &= 3 \end{aligned}$$

The first equation gives $t = -2s - 1$; plugging into the second equation gives $-7s = 7$, or $s = -1$. This results in $t = 1$. These values also satisfy the third equation, so the two lines intersect at the point $(3, -1, -2)$ as you can check.

Comments: Surprisingly most people got this wrong. Grading was binary: either you got full credit or no credit. This is a bit harsh so don't feel badly if you got a zero, but make sure you understand the solution. I'm not exactly sure what went wrong—perhaps some people thought that s would have to equal t at the intersection point, which is of course not true. This problem is a good test to make sure you understand what an equation of a line really is. It's also a nice introduction to solving multiple linear equations in multiple unknowns, if you've never seen that before. In 33A you'll learn systematic methods to solve such problems, which appear everywhere in math, science and engineering.

12. The directional vector for the given line is $(-1, -1, 3)$ a set of parametric equations for the line through $(1, 1, 3)$ would be $x = 1 - t, y = 1 - t, z - 3 + 3t$.

Page 39

1. An arbitrary point (x, y, z) of the plane satisfies $(x - 1, y + 2, z - 5) \cdot (3, -4, 1) = 0$, which means $3x - 3 - 4y - 8 + z - 5 = 0$ or $3x - 4y + z = 16$. To plot the plane, determine its intersections with the coordinate axes as shown in the text. The three intersection points are $(16/3, 0, 0)$, $(0, -4, 0)$ and $(0, 0, 16)$.

8. By inspection the normal vector of the second plane is $\mathbf{n} = (-1, 3, -4)$. Then the equation of the parallel plane through $(1, 2, 3)$ is $-x + 3y - 4z = \mathbf{n} \cdot (1, 2, 3) = -1 + 6 - 12 = -7$.

10. Normalize each equation so that the coefficient of y is 6 (the least common multiple of all the y coefficients). Then the normal vectors are $(-4, 6, 30)$, $(-4, 6, 30)$, $(6, -6, 2)$ and $(-4, 6, 30)$. The dot product of the first and third of these is $24 + 36 - 60 = 0$, so (c) is perpendicular to the other three. After normalization the right-hand side constants of (a), (b) and (d) are -24, 21 and -24 respectively, so (a) and (d) are coincident and (b) is parallel to them both.

12. The first equation gives $x = 2y - 3z + 5$, and plugging into the second equation results in $8(2y - 3z + 5) + 7y + z = 2$ or $23y - 23z = -38$, so let $z = t$ and then $y = t - 38/23$ and $x = -t + 5 - \frac{38 \cdot 2}{23} = -t + 39/23$. These are the scalar equations; the vector equation is then $\mathbf{x} = t(-1, 1, 1) + (39/23, -38/23, 0)$.

16. If the two intersected there would have to be some point $(x, y, z) = (3 + t, 1 - t, -2 + 3t)$ that satisfied $2x - y - z = 5$. Plugging the first into the second we get $2(3 + t) - (1 - t) - (-2 + 3t) = 5$, or $7 = 5$, which is impossible.

18. The line through P and Q is perpendicular to the plane and thus parallel to $\mathbf{N}$. It passes through $\mathbf{y}_0$ and so has equation $\mathbf{y}_0 + t\mathbf{N}$; since P is on the line we have $\overrightarrow{OP} = \mathbf{y}_0 + t_0\mathbf{N}$ for some specific t_0. Now P is also on the plane π, which has equation $\mathbf{N} \cdot (\mathbf{x} - \mathbf{x}_0) = 0$, so plug in $\overrightarrow{OP}$ for $\mathbf{x}$. This gives $\mathbf{N} \cdot (\mathbf{y}_0 + t_0\mathbf{N} - \mathbf{x}_0) = \mathbf{N} \cdot \mathbf{y}_0 + t_0 - \mathbf{N} \cdot \mathbf{x}_0 = 0$ (note that $\mathbf{N} \cdot \mathbf{N} = 1$) and the result follows.

    Comments: Although the solution is short this was certainly the hardest problem conceptually on the homework, and a lot of people had trouble with it. Try going through the solution again carefully and make sure you can follow it. Several people did things like take the dot product of a scalar with a vector, which makes no sense. Be sure to keep track of what is a vector and what is a scalar, and try to make sure each step you do makes sense rather than is some meaningless algebraic manipulation.