Math 32AH

Homework 4 Solutions

I graded 4 of the problems:
Page 120: 34, 36;
Page 126: 6, 10.

Each problem is worth 3 points. A grade of 0 indicates no solution or a substantially wrong solution. A grade of 3 indicates a correct or nearly correct solution. Otherwise the grade given is 1 or 2 depending upon how much work was put in and how close the solution is to being correct. How neat and clear your solution is also affects the grade I give.

If you believe a problem was misgraded, or I made some addition or other error, please write a short note explaining the situation, attach it to your homework, and return it to me (either in person, in my mailbox, or under my office door). I'll take a look and return the homework in the next section.

The following are solutions to the homework problems and additional comments for the problems I graded.

General Comments
The maximum number of points was 12. The high score was 12, the median was 11 and the mean was 10.1.

Page 120

2. Similar to the specific case $(0, 1]$ discussed in section, this is neither open nor closed. The sequence $\{a + \frac{1}{k}\}$ (the first few terms might be missing if they are greater than b) converges to a which is not in the set, so the set is not closed. On the other hand b is a boundary point and so the set is not open.

7. This sphere is not open because for each point if we draw an open ball around the point in $\mathbb{R}^3$ then the open ball contains points not on the sphere. On the other hand its complement is open: the complement consists of the open ball $x^2 + y^2 + z^2 < 9$ as well as an open set $x^2 + y^2 + z^2 > 9$.

8. This is the union of the set of points left of the vertical line $x = -1$ and right of the vertical line $x = 1$. The boundary of this set is the union of the two lines $x = -1$ and $x = 1$, and since none of these boundary points are in the set it follows the set is open. On the other hand these boundary points are all limit points of sequences of points in the set (for example $(0, 1)$ is the limit of $\{(0, 1 + \frac{1}{k})\}$) and so the set is not closed.

14. This is the set of points in the xy-plane outside the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$. If this were just a set of points in $\mathbb{R}^2$ then it would be open and not closed via a similar argument as in the solution of exercise 8 above. And in fact even in $\mathbb{R}^3$ there are still limit points of the set that are not in the set (namely points on the ellipse) and so the set is still not closed. However it is no longer open as well in $\mathbb{R}^3$, since now every point in the set is a boundary point. In other words if we draw a ball (a three-diminsional ball in this case), no matter how tiny, around any point in the set, it will contain points not in the set, namely all those outside the plane $z = 0$.

29. Solution is in the back of the book.

30. Just as in the first problem the first coordinate converges to 0 and the second is a constant 1, so the limit is $(0, 1)$.

34. We use l'Hôpital's rule for both coordinates. We have $\lim_{k\to\infty} k \sin\frac{1}{k} = \lim_{k\to\infty} \frac{\sin\frac{1}{k}}{\frac{1}{k}} = \lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1$ and $\lim_{k\to\infty} \frac{\ln k^2}{k} = \lim_{k\to\infty} \frac{2}{k} = 0$. So the limit is $(1, 0)$.

36. Although $\lim_{k\to\infty} e^{1/k} = e^0 = 1$ and $\lim_{k\to\infty} 1/k = 0$, $\lim_{k\to\infty} \ln\frac{1}{k} = \ln 0 = -\infty$ and so the sequence does not converge.

Page 126

2. There is no trouble with the denominator going to zero, so just plug in to get $\lim_{(x,y)\to(2,1)} \frac{x^3-y^3}{x^2-y^2} = \frac{8-1}{4-1} = \frac{7}{3}$.

6. This function is clearly continuous at all points aside from $(0,0)$, as it is a ratio of polynomials and the denominator is nonzero. It turns out the function is also continuous at $(0,0)$ as well. Using the method of the text, we have $|x| \leq |\mathbf{x}|$ and $|y| \leq |\mathbf{x}|$ where $\mathbf{x} = (x,y)$. It follows $\left|\frac{x^2y}{x^2+y^2}\right| = \frac{|x|^2|y|}{|\mathbf{x}|^2} \leq \frac{|\mathbf{x}|^2|\mathbf{x}|}{|\mathbf{x}|^2} = |\mathbf{x}|$. As $\mathbf{x} \to 0$ via any path we have $|\mathbf{x}| \to 0$, and so $\left|\frac{x^2y}{x^2+y^2}\right| \to 0$.

   Here's another method to solve the problem which is often more useful: translate into polar coordinates. So we have $x = r\cos\theta$ and $y = r\sin\theta$. Then $\frac{x^2y}{x^2+y^2} = \frac{r^3\cos^2\theta\sin\theta}{r^2} = r\cos^2\theta\sin\theta$. Now as $\mathbf{x} \to 0$ via any path we have $r \to 0$. Also $\cos^2\theta\sin\theta$ is always bounded between -1 and 1, so their product must go to zero, and the result follows.

10. As in exericise 6 the function is clearly continuous at all points aside from $(0,0)$. If we try out the polar coordinate method to see if it is continuous at $(0,0)$ we get $\frac{y^2(x-1)}{x^2+y^2} = \frac{r^2\sin^2\theta(r\cos\theta-1)}{r^2} = \sin^2\theta(r\cos\theta - 1)$. As $r \to 0$ this goes to $-\sin^2\theta$, so note that as we approach 0 from different angles we'll get different results (or even worse we could approach in a spiral!).

    Let's see how continuity fails more concretely. Try approaching zero along the positive x-axis, so that $y = 0$ and $x \to 0$ from the right. In this case we have $h(x,0) = \frac{0}{x^2} = 0$ for all $x > 0$, and so as $x \to 0$ we have $h(x,0) \to 0$. Now try approaching zero along the positive y-axis, so that $x = 0$ and $y \to 0$ from the top. In this case we have $h(0,y) = \frac{-y^2}{y^2} = -1$ for all $y > 0$, and so as $y \to 0$ we have $h(0,y) \to -1$. It follows h is not continuous at 0.

18. Both the numerator and denominator of f are continuous functions (sums and products of simpler continuous functions). The denominator also can never equal zero. Therefore f is continuous.