Math 32AH
Homework 7 Solutions

I graded 4 of the problems:
Page 178: 14;
Page 187: 8;
Page 193: 16;
Page 201: 16.

Each problem is worth 3 points. A grade of 0 indicates no solution or a substantially wrong solution. A grade of 3 indicates a correct or nearly correct solution. Otherwise the grade given is 1 or 2 depending upon how much work was put in and how close the solution is to being correct. How neat and clear your solution is also affects the grade I give.

If you believe a problem was misgraded, or I made some addition or other error, please write a short note explaining the situation, attach it to your homework, and return it to me (either in person, in my mailbox, or under my office door). I'll take a look and return the homework in the next section.

The following are solutions to the homework problems and additional comments for the problems I graded.

General Comments
The maximum number of points was 12. The high score was 12, the median was 9 and the mean was 8.9.

Page 178

2. We have $D_u(f(x_0)) = (4,-1)\cdot(1/\sqrt{2}, 1/\sqrt{2}) = 3/\sqrt{2}$.

6. We have $D_u(f(x_0)) = (8,-8,6)\cdot(2/3,2/3,1/3) = 2$.

11. Solution is in the back of the book.

13. Solution is in the back of the book.

14. We have $\nabla g(1,1,2) = (-\frac{1}{9}, -\frac{2}{9}, -\frac{2}{9})$, and the unit vector in this direction is $(-\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3})$. Note that this is the same answer as for problem 13.

Page 187

2. The general formula is $(t\sqrt{t^2+t^4}, t^2\sqrt{t^2+t^4})\cdot(1,2t) = t\sqrt{t^2+t^4} + 2t^2\sqrt{t^2+t^4}$. Evaluating at $t=-2$ we get $60\sqrt{65}$.

6. The general formula is $(\frac{t}{\sqrt{t^2+t^4+t^6}}, \frac{t^2}{\sqrt{t^2+t^4+t^6}}, \frac{t^3}{\sqrt{t^2+t^4+t^6}})\cdot(1,2t,3t^2) = \frac{1}{\sqrt{t^2+t^4+t^6}}(t+2t^3+3t^5)$. Evaluating at $t=-1$ we get $-2\sqrt{3}$.

8. Comments: A common error was to parameterize the line segment as $(2,1,1)+t(3,2,-1)$. I made the same mistake! I was confusing the parameterization we do in this class $\mathbf{a}+t(\mathbf{b}-\mathbf{a})$ with another one typically used which is $(1-t)\mathbf{a}+t\mathbf{b}$. You can easily see that these are the same, but be careful you don't use part of one and part of the other!

   We want to find a point $c=(x,y,z)$ on the line segment $(2,1,1)+t(1,1,-2)$ $(0 \le t \le 1)$ such that $8 = (2x,2y,2z)\cdot(1,1,-2) = 2x+2y-4z$. Plugging in we get
   $4 = (2+t)+(1+t)+(-2+4t) = 1+6t$, and so $t=\frac{1}{2}$. Therefore $c = (\frac{5}{2}, \frac{3}{2}, 0)$.

11. Solution is in the back of the book. For some reason they didn't substitute the values of x,y,z and simplify things.

18. The total derivative of w is
   $(\frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}, \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}) = ((2xy+x)(te^{st}) + (x^2+x)(2s), (2xy+x)(se^{st}) + (x^2+x)(2t))$.
   Evaluating at $(2,1)$ we get $(15e^4+e^2, 24e^4+2e^2)$.

24. We have $\nabla f(x,y,z) = (y^2+3x^2, 2xy+z^2, 2yz+3z^2)$, and evaluating at $(0,0,4)$ gives $(0,16,48)$; we can use any multiple so let's pick $(0,1,3)$. So the tangent plane has equation $(0,1,3)\cdot(x,y,z-4)=0$ or $y+3z=12$.

32. (a) We have $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$ and $\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = -\frac{\partial z}{\partial x}r\sin\theta + \frac{\partial z}{\partial y}r\cos\theta$.

(b) We have $(\frac{\partial z}{\partial r})^2 = (\frac{\partial z}{\partial x})^2\cos^2\theta + (\frac{\partial z}{\partial y})^2\sin^2\theta + 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\cos\theta\sin\theta$ and $\frac{1}{r^2}(\frac{\partial z}{\partial \theta})^2 = (\frac{\partial z}{\partial x})^2\sin^2\theta + (\frac{\partial z}{\partial y})^2\cos^2\theta - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\cos\theta\sin\theta$ and the result follows.

Page 193

3. Solution is in the back of the book.

5. Solution is in the back of the book.

8. We have $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{1-yz\cos xyz}{1-xy\cos xyz}$ and $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{1-xz\cos xyz}{1-xy\cos xyz}$; these are valid whenever $1 - xy\cos xyz \neq 0$.

13. Solution is in the back of the book.

16. Let's try solving this both ways. One equation is $\nabla F(2,3,6)\cdot(x-2, y-3, z-6) = 0$ implies $(4,6,12)\cdot(x-2,y-3,z-6) = 0$ implies $(2,3,6)\cdot(x-2,y-3,z-6) = 0$ implies $2x+3y+6z = 49$. The other way is to calculate $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{x}{z} = -\frac{1}{3}$ and $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{y}{z} = -\frac{1}{2}$, and then the equation is $z - 6 = -\frac{1}{3}(x-2) - \frac{1}{2}(y-3)$ which is also $2x+3y+6z = 49$.

21. Solution is in the back of the book. Note that you use the method of problem 19 to solve this.

Page 201

5. Solution is in the back of the book.

6. We have $f_x = f_y = (x+y)^{-1}$, $f_{xx} = f_{xy} = f_{yx} = f_{yy} = -(x+y)^{-2}$, and all third partial derivatives are $2(x+y)^{-3}$.

13. We have $f_x = \frac{2x}{x^2+y^2}$ and $f_{xx} = \frac{2(y^2-x^2)}{(x^2+y^2)^2}$; also $f_y = \frac{2y}{x^2+y^2}$ and $f_{yy} = \frac{2(x^2-y^2)}{(x^2+y^2)^2}$. Adding them gives 0.

16. From problem 32(a) of page 187, we have $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$ and $\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = -\frac{\partial z}{\partial x}r\sin\theta + \frac{\partial z}{\partial y}r\cos\theta$.

Now $\frac{\partial^2 z}{\partial r^2} = \frac{\partial^2 z}{\partial x\partial r}\cos\theta + \frac{\partial^2 z}{\partial y\partial r}\sin\theta$ by using equality of mixed partials (be sure you understand where that was used). Now we have

$$\begin{aligned}\frac{\partial^2 z}{\partial x\partial r} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}\right)\\ &= \frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial r\partial x} + \frac{\partial^2 z}{\partial x\partial y}\frac{\partial y}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial r\partial x}\\ &= \frac{\partial^2 z}{\partial x^2}\cos\theta + \frac{\partial^2 z}{\partial x\partial y}\sin\theta.\end{aligned}$$

Similarly $\frac{\partial^2 z}{\partial y\partial r} = \frac{\partial^2 z}{\partial y^2}\sin\theta + \frac{\partial^2 z}{\partial x\partial y}\cos\theta$, and it follows $\frac{\partial^2 z}{\partial r^2} = \frac{\partial^2 z}{\partial x^2}\cos^2\theta + 2\frac{\partial^2 z}{\partial x\partial y}\cos\theta\sin\theta + \frac{\partial^2 z}{\partial y^2}\sin^2\theta$.

Similarly tedious calculations (but be sure to do them all, as they are good practice in getting comfortable with partial differentiation) yield $\frac{\partial^2 z}{\partial \theta^2} = \frac{\partial^2 z}{\partial x^2}r^2\sin^2\theta - \frac{\partial z}{\partial x}r\cos\theta - \frac{\partial z}{\partial y}r\sin\theta - 2\frac{\partial^2 z}{\partial x\partial y}r^2\cos\theta\sin\theta + \frac{\partial^2 z}{\partial y^2}r^2\cos^2\theta$.

We thus have $\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial z}{\partial r} = (\frac{\partial^2 z}{\partial x^2}\cos^2\theta + 2\frac{\partial^2 z}{\partial x\partial y}\cos\theta\sin\theta + \frac{\partial^2 z}{\partial y^2}\sin^2\theta) + (\frac{\partial^2 z}{\partial x^2}\sin^2\theta - \frac{\partial z}{\partial x}\frac{1}{r}\cos\theta - \frac{\partial z}{\partial y}\frac{1}{r}\sin\theta - 2\frac{\partial^2 z}{\partial x\partial y}\cos\theta\sin\theta + \frac{\partial^2 z}{\partial y^2}\cos^2\theta) + (\frac{\partial z}{\partial x}\frac{1}{r}\cos\theta + \frac{\partial z}{\partial y}\frac{1}{r}\sin\theta) = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$ as desired.