Math 32AH
Homework 8 Solutions

I graded 4 of the problems:
Page 210: 32;
Page 218: 4,8;
Page 228: 17;

Each problem is worth 3 points. A grade of 0 indicates no solution or a substantially wrong solution. A grade of 3 indicates a correct or nearly correct solution. Otherwise the grade given is 1 or 2 depending upon how much work was put in and how close the solution is to being correct. How neat and clear your solution is also affects the grade I give.

If you believe a problem was misgraded, or I made some addition or other error, please write a short note explaining the situation, attach it to your homework, and return it to me (either in person, in my mailbox, or under my office door). I'll take a look and return the homework in the next section.

The following are solutions to the homework problems and additional comments for the problems I graded.

General Comments
The maximum number of points was 12. The high score was 12, the median was 11 and the mean was 9.4.

Page 210

26. Let $f(x,y,z) = \cos(x+y+z)$. Then $f'(x,y,z) = (-\sin(x+y+z), -\sin(x+y+z), -\sin(x+y+z))$ and

$$f''(x,y,z) = \begin{pmatrix} -\cos(x+y+z) & -\cos(x+y+z) & -\cos(x+y+z) \\ -\cos(x+y+z) & -\cos(x+y+z) & -\cos(x+y+z) \\ -\cos(x+y+z) & -\cos(x+y+z) & -\cos(x+y+z) \end{pmatrix}.$$

We have $f'(\pi/6,\pi/6,\pi/6) = (-1,-1,-1)$ while $f''(\pi/6,\pi/6,\pi/6)$ is a 3×3 matrix all of whose entries is 0. It follows the second degree Taylor polynomial is simply $f(x,y,z) = \pi/2 - x - y - z$.

32. Let $f(x,y,z) = \sqrt{x^2+y^2+z^2}$. Then $f'(x,y,z) = \left(\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}\right)$ and

$$f''(x,y,z) = \begin{pmatrix} \frac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}} & \frac{-xy}{(x^2+y^2+z^2)^{3/2}} & \frac{-xz}{(x^2+y^2+z^2)^{3/2}} \\ \frac{-xy}{(x^2+y^2+z^2)^{3/2}} & \frac{x^2+z^2}{(x^2+y^2+z^2)^{3/2}} & \frac{-yz}{(x^2+y^2+z^2)^{3/2}} \\ \frac{-xz}{(x^2+y^2+z^2)^{3/2}} & \frac{-yz}{(x^2+y^2+z^2)^{3/2}} & \frac{x^2+y^2}{(x^2+y^2+z^2)^{3/2}} \end{pmatrix}.$$

We then have $f(2,2,1) = 3$, $f'(2,2,1) = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$, and

$$f''(2,2,1) = \frac{1}{27}\begin{pmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{pmatrix}.$$

Letting $\mathbf{x} = (2.01, 1.94, 0.98)$, we have $\mathbf{x} - (2,2,2) = (0.01, -0.06, -0.02)$, and we have

$$f(\mathbf{x}) \approx 3 - \frac{2}{3}\frac{6}{100} + \frac{225}{2\cdot 27\cdot 100\cdot 100} = 2.96 + \frac{1}{2400} \approx 2.96042$$

and this is a very good approximation as you can check.

Page 218

3. We have $f'(x,y) = (2x-y, -x-2y+5)$. If this is $(0,0)$, then $(x,y) = (1,2)$. Since $f_{xx} = 2$, $f_{yy} = -2$ and $f_{xy} = -1$, $D = -3$ and the point is a saddle point.

4. We have $f'(x,y) = (3x^2+6x, 3y^2-6y)$. There are thus four critical points $(0,0), (-2,0), (0,2), (-2,2)$. Since $f_{xx} = 6x+6$, $f_{yy} = 6y-6$ and $f_{xy} = 0$, $D = 36(x+1)(y-1)$. It follows $(0,0)$ and $(-2,2)$ are saddle points, $(-2,0)$ is a local maximum, and $(0,2)$ is a local minimum.

8. We have $f'(x,y,z) = (-2x, -4y+z, -2z+y)$. The sole critical point is at $(0,0,0)$. We have

$$f''(2,2,1) = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -4 & 1 \\ 0 & 1 & -2 \end{pmatrix},$$

and by Theorem 9.10 it follows $(0,0,0)$ is a local maximum.

11. Solution is in the back of the book. We are trying to maximize xyz where $xy + 2xz + 2yz = 12$.

25. The derivative method doesn't work here but we can see that $f(x,y) = (\sqrt{x} - \sqrt{y})^2 \geq 0$ for all $x, y \geq 0$, and since $f(x,y) = 0$ whenever $x = y$ this minimum is achieved.

Page 228

2. We have $\nabla f = (1,1)$ and $\nabla g = (18x, 8y)$. It follows $1 = 18x\lambda$ and $1 = 8y\lambda$. Therefore $\frac{9}{(18\lambda)^2} + \frac{4}{(8\lambda)^2} = 36$, and solving we get $\lambda = \pm\frac{\sqrt{13}}{72}$. The extreme values are thus $\pm\left(\frac{4}{\sqrt{13}} + \frac{9}{\sqrt{13}}\right)$.

4. We have $\nabla f = (y,x)$ and $\nabla g = (2x, 2y)$. It follows $y = 2x\lambda$ and $x = 2y\lambda$; eliminating λ we get $x^2 = y^2$, and plugging into g gives $x, y = \pm 2$. The extrema are thus ± 4.

11. We have $\nabla f = (2,1,2)$, $\nabla g_1 = (2x, 2y, 0)$ and $\nabla g_2 = (1,0,1)$. It follows $2 = 2x\lambda_1 + \lambda_2$, $1 = 2y\lambda_1$ and $2 = \lambda_2$. From these we deduce that $x = 0$ (we cannot have $\lambda_1 = 0$ since that would violate the second equation), and then from the constraints we know that $y = \pm 2$ and $z = 2$. The extrema are thus 2 and 6. (Same solution is in the back of the book.)

15. Solution is in the back of the book.

17. Solution is in the back of the book. For the method see solution 22 below.

20. We find the extrema of the square of the distance from a point of the ellipsoid to the origin $f(x,y,z) = x^2 + y^2 + z^2$ where the constraint is $g(x,y,z) = 4x^2 + 9y^2 + 36z^2 = 1$. Then $\nabla f = (2x, 2y, 2z)$, $\nabla g = (8x, 18y, 72z)$, and it follows $x = 4x\lambda$, $y = 18y\lambda$, and $z = 72z\lambda$. The only way all three of these can be satisfied is if two of x, y, z are zero. It follows the minimum distance occurs at point $(0,0,1)$ (distance 1) and the maximum distance occurs at point $(3,0,0)$ (distance 3).

22. By symmetry we can choose the sides of the parallelepiped to be parallel to the coordinate axes, and also by symmetry once we pick one point (x,y,z) (say in the all positive quadrant) the other points of the parallelepiped are all determined, and the volume is $(2x)(2y)(2z) = 8xyz$. It suffices to maximize $f(x,y,z) = xyz$ where $g(x,y,z) = x^2 + y^2 + z^2 = a$. Then $\nabla f = (yz, xz, xy)$, $\nabla g = (2x, 2y, 2z)$, and it follows $yz = 2x\lambda$, $xz = 2y\lambda$, and $xy = 2z\lambda$. If $\lambda \neq 0$ then we have $\frac{yz}{x} = \frac{xz}{y}$, if $z \neq 0$ we have $x^2 = y^2$. Also $\frac{yz}{x} = \frac{xy}{z}$, and if $y \neq 0$ then $x^2 = z^2$. Since geometrically we're requiring x, y, z to all be positive we see that $x = y = z$ satisfies all of these equalities (with $\lambda = x/2$), and plugging into g we get $(x,y,z) = \left(\sqrt{\frac{a}{3}}, \sqrt{\frac{a}{3}}, \sqrt{\frac{a}{3}}\right)$. This must be a maximum since you can check that $x = y = -z$ gives a minimum (and is also geometrically impossible in our situation). As we might expect the parallelepiped of largest volume is an inscribed cube. Note that $\lambda = 0$ or $xyz = 0$ give values which satisfy the equations but are not extrema (although they do happen to give rise to some but not all of the minimum volume solutions of the geometric problem).