Math 32AH
Midterm 1 Solutions
October 29, 2004

1. (20 points) Let $A = (1, 0, 1)$, $B = (-1, 2, 1)$, $C = (2, 1, 0)$ and $D = (2, 2, 2)$ be four points in $\mathbb{R}^3$.

   (a) (4 points) Find an equation for the line through A and B.
   One possible equation is $A + t(B - A) = (1, 0, 1) + t(-2, 2, 0)$ for $t \in \mathbb{R}$.

   (b) (4 points) Determine the angle between the vectors $\overrightarrow{AC}$ and $\overrightarrow{AD}$.
   The angle is $\theta = \cos^{-1} \frac{(C-A)\cdot(D-A)}{|C-A||D-A|} = \cos^{-1} \frac{2}{\sqrt{3}\sqrt{6}} = \cos^{-1} \frac{2}{3\sqrt{2}}$.

   (c) (4 points) Find an equation for the plane π through A, B and C.
   First construct the normal vector as $(B - A) \times (C - A) = (-2, 2, 0) \times (1, 1, -1) = (-2, -2, -4)$. Any multiple of this is also a normal vector so we could choose $(1, 1, 2)$ to be cleaner. Then a point (x, y, z) is on the plane if and only if $(x - 1, y, z - 1) \cdot (1, 1, 2) = 0$, or $x + y + 2z = 3$.

   (d) (4 points) Find the distance from D to the plane π.
   Project $D - A = (1, 2, 1)$ onto the unit normal vector $\mathbf{N} = \frac{1}{\sqrt{6}}(1, 1, 2)$ and take the length. The projection is $((D - A) \cdot \mathbf{n})\mathbf{N} = \frac{5}{6}(1, 1, 2)$, and so the length is $\frac{5}{\sqrt{6}}$ which is the distance from D to π. Note that this is essentially the same as using the formula in exercise 18 on page 39.

   (e) (4 points) Find a line passing through D and perpendicular to the plane π.
   From part (c) a normal vector to the plane is $(1, 1, 2)$. So one possibile equation for the line would be $(2, 2, 2) + t(1, 1, 2)$ for $t \in \mathbb{R}$.

2. (20 points) Note: If you've forgotten the answer to part (a) or part (b), you may "buy" the answer from the teaching assistant for 2 points (each). Just come up to the front and ask.

   (a) (2 points) State the Cauchy-Schwarz inequality (you do not need to prove it).
   $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}|\,|\mathbf{y}|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

   (b) (2 points) State the triangle inequality.
   $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

   (c) (6 points) Prove the triangle inequality using the Cauchy-Schwarz inequality.

$$\begin{aligned} |\mathbf{x} + \mathbf{y}|^2 &= (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) \\ &= |\mathbf{x}|^2 + 2\mathbf{x} \cdot \mathbf{y} + |\mathbf{y}|^2 \\ &\leq |\mathbf{x}|^2 + 2\,|\mathbf{x}|\,|\mathbf{y}| + |\mathbf{y}|^2 \\ &= (|\mathbf{x}| + |\mathbf{y}|)^2 \end{aligned}$$

   Now take square roots of both sides.

   (d) (5 points) Assuming the identity $|\mathbf{v} \times \mathbf{w}|^2 + (\mathbf{v} \cdot \mathbf{w})^2 = |\mathbf{v}|^2\,|\mathbf{w}|^2$ is true, prove

$$|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}|\,|\mathbf{w}| \sin\theta,$$

   where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$.

$$\begin{aligned} |\mathbf{v} \times \mathbf{w}|^2 &= |\mathbf{v}|^2\,|\mathbf{w}|^2 - (\mathbf{v} \cdot \mathbf{w})^2 \\ &= |\mathbf{v}|^2\,|\mathbf{w}|^2 - (|\mathbf{v}|\,|\mathbf{w}| \cos\theta)^2 \\ &= |\mathbf{v}|^2\,|\mathbf{w}|^2 (1 - \cos^2\theta) \\ &= |\mathbf{v}|^2\,|\mathbf{w}|^2 \sin^2\theta \end{aligned}$$

   Now take square roots of both sides.

(e) (5 points) Prove that the area of the parallelogram formed by $\mathbf{v}$ and $\mathbf{w}$ is $|\mathbf{v} \times \mathbf{w}|$. You may use the result of part (d) in your proof, even if you are unable to prove part (d).

The area of the parallelogram is base times height. Letting $\mathbf{v}$ be the base, its length is $|\mathbf{v}|$. The height is then $|\mathbf{w}| \sin \theta$ (drawing a picture here is good), and the result follows from part (d).

3. (20 points) Let $\mathbf{x}(t) = (a \cos t, a \sin t, bt)$ where a and b are real numbers.

(a) (10 points) Find the arc length of $\mathbf{x}(t)$ from $t = 0$ to $t = 2\pi$.

The arc length is $\int_0^{2\pi} |\dot{\mathbf{x}}(t)|\, dt = \int_0^{2\pi} \sqrt{a^2 + b^2} dt = 2\pi\sqrt{a^2 + b^2}$.

(b) (10 points) Parameterize $\mathbf{x}(t)$ by arc length.

The speed is $\sqrt{a^2 + b^2}$ so let $t(s) = \frac{s}{\sqrt{a^2+b^2}}$. Then $\mathbf{x}(t(s)) = (a \cos \frac{s}{\sqrt{a^2+b^2}}, a \sin \frac{s}{\sqrt{a^2+b^2}}, \frac{bs}{\sqrt{a^2+b^2}}) = \mathbf{y}(s)$ is parameterized by arc length.

4. (20 points) Let $\mathbf{x}(t) = (a \cos t, a \sin t, bt)$ where a and b are real numbers. Compute the following.

(a) (4 points) K

$K = \frac{|\dot{\mathbf{T}}(t)|}{|\dot{\mathbf{x}}(t)|} = \frac{|(-a \cos t, -a \sin t, 0)|}{a^2+b^2} = \frac{|a|}{a^2+b^2}$ if a, b are not both zero.

(b) (4 points) $\mathbf{T}$

$\mathbf{T} = \frac{\dot{\mathbf{x}}(t)}{|\dot{\mathbf{x}}(t)|} = \frac{(-a \sin t, a \cos t, b)}{\sqrt{a^2+b^2}}$ if a, b are not both zero.

(c) (4 points) $\mathbf{N}$

$\mathbf{N} = \frac{\dot{\mathbf{T}}(t)}{|\dot{\mathbf{T}}(t)|} = \frac{(-a \cos t, -a \sin t, 0)}{|a|}$ if a is not zero.

(d) (4 points) $\mathbf{a_T}$

$\mathbf{a_T} = (\mathbf{a} \cdot \mathbf{T})\mathbf{T} = ((-a \cos t, -a \sin t, 0) \cdot \frac{(-a \sin t, a \cos t, b)}{\sqrt{a^2+b^2}}) \frac{(-a \sin t, a \cos t, b)}{\sqrt{a^2+b^2}} = \mathbf{0}$.

(e) (4 points) $\mathbf{a_N}$

$\mathbf{a_N} = \mathbf{a} - \mathbf{a_T} = (-a \cos t, -a \sin t, 0)$.

5. (20 points) Determine whether or not each of the following functions is continuous at $(0, 0)$. Prove your answer.

(a) (10 points)

$$f(x, y) = \begin{cases} \frac{x^3}{x^3+y} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

This function is not continuous at $(0, 0)$. Fix $x = 0$ and have y approach zero from the top. Then $\lim_{y\to0^+} f(0, y) = \lim_{y\to0^+} \frac{0}{y} = 0$. This is okay, but now now fix $y = 0$ and have x approach zero from the right. Then $\lim_{x\to0^+} f(x, 0) = \lim_{x\to0^+} \frac{x^3}{x^3} = \lim_{x\to0^+} 1 = 1$. Since the limit must 0 when approaching $(0, 0)$ from any direction, it means the function cannot be continuous.

(b) (10 points)

$$g(x, y) = \begin{cases} \frac{xy^2 \sin(xy)}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

This function is continuous at $(0, 0)$. Use polar coordinates, and let $x = r \cos \theta$ and $y = r \sin \theta$. Then $g(x, y) = h(r, \theta) = \frac{r^3 \cos \theta \sin^2 \theta \sin(r^2 \cos \theta \sin \theta)}{r^2} = r \cos \theta \sin^2 \theta \sin(r^2 \cos \theta \sin \theta)$. Now

$$\cos \theta \sin^2 \theta \sin(r^2 \cos \theta \sin \theta)$$

is bounded between -1 and $+1$ for all values of r and θ, and $r \to 0$ as $(x, y) \to (0, 0)$. Therefore $\lim_{(x,y)\to(0,0)} g(x, y) = \lim_{r\to0} h(r, \theta)$ exists and is equal to zero.