Math 32AH
Midterm 2 Solutions
November 24, 2004

1. (20 points) Let $f: \mathbf{R}^2 \to \mathbf{R}$ be a linear function in two variables.
   (a) (10 points) Show that f must have the form of $f(x,y) = ax + by$ where a, b are two real numbers.
   Since f is linear we have $f(x,y) = f(x,0) + f(0,y) = xf(1,0) + yf(0,1)$. Now let $a = f(1,0)$ and $b = f(0,1)$.
   (b) (5 points) Show that f is continuous at all points of $\mathbf{R}^2$ from the definition of continuity.
   See the proof of Theorem 2.6 of Chapter 4 of the text.
   (c) (5 points) Compute the total derivative of f.
   We have $D_f(x,y) = \nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (a,b)$.

2. (20 points)
   (a) (10 points) Let $f(x,y) = x^2 - 4y^2 - 3xy$, compute the total derivative f' and the Hessian f".
   We have $D_f(x,y) = \nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (2x - 3y, -8y - 3x)$., and so $f'' = \begin{pmatrix} 2 & -3 \\ -3 & -8 \end{pmatrix}$.
   (b) (10 points) Let $f = (x^2 + y^2 + z^2)^{\frac{1}{2}}$, prove that f satisfies the equation:
   $$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{2}{f}$$
   We have $\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}}$ and $\frac{\partial^2 f}{\partial x^2} = \frac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}}$. By symmetry we have $\frac{\partial^2 f}{\partial y^2} = \frac{x^2+z^2}{(x^2+y^2+z^2)^{3/2}}$ and $\frac{\partial^2 f}{\partial z^2} = \frac{x^2+y^2}{(x^2+y^2+z^2)^{3/2}}$. Their sum is then $\frac{2}{\sqrt{x^2+y^2+z^2}}$.

3. (20 points) Let $f(x,y)$ be a two variable function.
   (a) (10 points) Assume $\nabla f(x_0,y_0) \neq 0$, show that $\nabla f(x_0,y_0)$ is the maximum increasing direction of f.
   Given a direction $\mathbf{u}$, we have $\frac{\partial f}{\partial u}(x_0,y_0) = \nabla f(x_0,y_0) \cdot \mathbf{u} = |\nabla f(x_0,y_0)|\,|\mathbf{u}| \cos\theta$, where θ is the angle between the vectors $\nabla f(x_0,y_0)$ and $\mathbf{u}$. The maximum value is achieved when $\cos\theta = 1$, which occurs when $\theta = 0$, so that $\mathbf{u}$ is the direction of $\nabla f(x_0,y_0)$.
   (b) (10 points) Assume f has a local extreme value at (x_0,y_0), then show that $\nabla f(x_0,y_0) = 0$.
   See the proof of Theorem 9.3 of Chapter 4 of the text. Although we haven't reached this section of the text the proof was covered in lecture.

4. (20 points) What is the equation of the tangent line to the curve of the intersection of the two surfaces
   $$x^2 + y^2 - z = 8, \quad x - y^2 + z^2 = -2$$
   at the point $(2,-2,0)$.
   Using the method of Exercise 19 of page 194, the tangent vector is $(2x, 2y, -1) \times (1, -2y, 2z)$ evaluated at $(2,-2,0)$ which is $(4,-4,-1) \times (1,4,0) = (4,-1,20)$. The equation of the tangent line is then $(2,-2,0) + t(4,-1,20)$.

5. (20 points) Let $w = x^2 + y^2 + z^2$, and $\mathbf{x} = (x,y,z) = (s^2 + t^2 + u^2, stu, s^2 + t^2u)$. Find the total derivative of $w = f(\mathbf{x}(s,t,u))$.
   We have $D_f(x,y,z) = (2x, 2y, 2z)$, and in matrix notation we have
   $$D_{\mathbf{x}}(s,t,u) = \begin{pmatrix} 2s & 2t & 2u \\ tu & su & st \\ 2s & 2tu & t^2 \end{pmatrix}.$$
   Multiplying the two gives $(4sx + 2tuy + 4sz, 6tx + 2suy + 4tuz, 4ux + 2sty + 4t^2z)$; you can then evaluate at $(x,y,z) = (s^2 + t^2 + u^2, stu, s^2 + t^2u)$ to get the answer purely in terms of s, t, u.