Math 32BH
Final Solutions
March 22, 2005

1. (20 points) Apply Stokes theorem to evaluate the surface integral

$$\int\int_S \operatorname{curl}\overrightarrow{\mathbf{F}} \cdot d\overrightarrow{S}$$

where $\overrightarrow{\mathbf{F}} = (3z, 5x, -2y)$ and S is part of the surface $z = 2x^2 + 2y^2$ below the plane $z = 8$ and whose orientation is given by the upper unit normal vector.

Parameterize the boundary $8 = 2x^2+2y^2$ as $(2\cos t, 2\sin t, 8)$, and so a normal vector is $(-2\sin t, 2\cos t, 0)$. Note however that in order to get the correct answer with Stokes' theorem you need to use a **unit** normal vector, and so we must use $(-\sin t, \cos t, 0)$. Everyone missed this fact (which is not emphasized properly in the textbook), including me! Our unit normal vector points up as can be seen by evaluating it at $t = 0$. Using Stokes' theorem we get $\int\int_S \operatorname{curl}\overrightarrow{\mathbf{F}} \cdot d\overrightarrow{S} = \int_0^{2\pi} (24, 10\cos t, -4\sin t) \cdot (-\sin t, \cos t, 0)\, dt = \int_0^{2\pi} -24\sin t + 10\cos^2 t\, dt = 10\pi$.

2. (20 points) Compute the improper integral

$$\int\int\int_{\mathbb{R}^3} \sqrt{x^2+y^2+z^2}\, e^{-\sqrt{x^2+y^2+z^2}} dV.$$

Paremeterizing $\mathbb{R}^3$ with spherical coordinates we get $\int\int\int_{\mathbb{R}^3} e^{-\sqrt{x^2+y^2+z^2}} dV = \int_0^\infty \rho^3 e^{-\rho} \int_0^\pi \sin\phi \int_0^{2\pi} \, d\theta d\phi d\rho = 24\pi$.

3. (20 points) Compute the surface integral $\int_S \overrightarrow{F} \cdot d\overrightarrow{S}$ where

$$\overrightarrow{F}(x, y, z) = \frac{\overrightarrow{x}}{|\overrightarrow{x}|^3}$$

with $\overrightarrow{x} = (x, y, z)$ and S is the surface

$$\frac{x^2}{2^2} + \frac{y^2}{3^2} + \frac{z^2}{4^2} = 1$$

with outward orientation.

This is Example 8.5 on page 485 of the text, with $k = 1$. The solution is 4π.

4. (20 points)

(a) (10 points) Give a precise geometric description of each of the cylindrical coordinates of a point $P \in \mathbb{R}^3$. Express each cylindrical coordinate in terms of the rectangular coordinates, noting any conditions that might apply. Express each rectangular coordinate in terms of the cylindrical coordinates, noting any conditions that might apply. Describe the restrictions you would place on the cylindrical coordinates so that every point in $\mathbb{R}^3$ has at least one representation in cylindrical coordinates, and as many points as possible have exactly one representation in cylindrical coordinates. Describe precisely the set of points that has more than one representation under your restrictions, and describe the set of all representations for each such point.

The cylindrical coordinates of P are (r, θ, z), where r $(r \geq 0)$ is the distance from P' (the projection of P onto the xy-plane) to the origin, θ is the angle $(\theta \in [0, 2\pi))$ that the line segment from the origin to P' makes with the positive x-axis, and z is the distance of the point to the xy plane. We have $r = \sqrt{x^2 + y^2}$, $z = z$, and $\theta = \tan^{-1} \frac{y}{x}$ for $x > 0$ and $y \geq 0$; $\theta = \tan^{-1} \frac{y}{x} + 2\pi$ for $x > 0$ and $y < 0$; and $\theta = \tan^{-1} \frac{y}{x} + \pi$ for $x < 0$. If $x = 0$ then $\theta = \pi/2$ if $y > 0$ and $\theta = 3\pi/2$ if $y < 0$; it is undefined if $y = 0$, see below. Conversely $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$. The restrictions on r and θ already mentioned ensure every point not on the z axis has exactly one cylindrical coordinate tuple. Those points can be represented as $(0, \theta, z)$ for any $\theta \in [0, 2\pi)$.

(b) (10 points) Give a precise geometric description of each of the spherical coordinates of a point $P \in \mathbb{R}^3$. Express each spherical coordinate in terms of the rectangular coordinates, noting any conditions that might apply. Express each rectangular coordinate in terms of the spherical coordinates, noting any conditions that might apply. Describe the restrictions you would place on the spherical coordinates so that every point in $\mathbb{R}^3$ has at least one representation in spherical coordinates, and as many points as possible have exactly one representation in spherical coordinates. Describe precisely the set of points that has more than one representation under your restrictions, and describe the set of all representations for each such point.

The spherical coordinates of P are (ρ, ϕ, θ), where ρ $(\rho \geq 0)$ is the distance from P to the origin, ϕ $(\phi \in [0, \pi])$ is the angle the line segment from the origin to P makes with the positive z axis, and θ is the angle $(\theta \in [0, 2\pi))$ that the line segment of the projection of P onto the xy plane makes with the x-axis. We have $\rho = \sqrt{x^2 + y^2 + z^2}$, $\phi = \cos^{-1} \frac{z}{\sqrt{x^2+y^2+z^2}}$ (for $(x, y, z) \neq (0, 0, 0)$; otherwise undefined as noted below) and $\theta = \tan^{-1} \frac{y}{x}$ for $x > 0$ and $y \geq 0$; $\theta = \tan^{-1} \frac{y}{x} + 2\pi$ for $x > 0$ and $y < 0$; and $\theta = \tan^{-1} \frac{y}{x} + \pi$ for $x < 0$. If $x = 0$ then $\theta = \pi/2$ if $y > 0$ and $\theta = 3\pi/2$ if $y < 0$; it is undefined if $y = 0$, see below. Conversely $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$. The restrictions on ρ, ϕ and θ already mentioned ensure every point not on the z axis has exactly one spherical coordinate tuple. All points on the positive z axis can be represented as $(z, 0, \theta)$ for any $\theta \in [0, 2\pi)$. All points on the negative z axis can be represented as $(-z, \pi, \theta)$ for any $\theta \in [0, 2\pi)$. The origin can be represented as $(0, \phi, \theta)$ for any $\phi \in [0, \pi]$ and any $\theta \in [0, 2\pi)$.

5. (20 points) Find the surface area cut from the paraboloid $z = x^2 + y^2$ by the cylinder $x^2 + y^2 \leq 1$.

Parameterize the surface S by $(u\cos v, u\sin v, u^2)$, where $(u, v) \in [0, 1] \times [0, 2\pi]$. Then $X_u = (\cos v, \sin v, 2u)$ and $X_v = (-u\sin v, u\cos v, 0)$. It follows $X_u \times X_v = (-2u^2\cos v, -2u^2\sin v, u)$, and $|X_u \times X_v| = u\sqrt{1 + 4u^2}$. The area is thus $\int_0^{2\pi} \int_0^1 u\sqrt{1 + 4u^2}\, du\, dv = \frac{\pi}{6}\left(5\sqrt{5} - 1\right)$.

6. (20 points) Evaluate the integral $\int\int\int_C \sqrt{x^2 + y^2 + z^2} dV$ where C is the ice cream cone

$$\{(x, y, z) | x^2 + y^2 + z^2 \leq 1, x^2 + y^2 \leq z^2, z \geq 0\}.$$

This is similar to Example 8.7 in the text, except $\phi \in [0, \pi/4]$. The result is thus $\int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho^3 \sin\phi\, d\rho\, d\phi\, d\theta = \frac{\pi}{2}\left(1 - \frac{1}{\sqrt{2}}\right)$.

7. (20 points)

(1) If the function $f(x, y, z)$ has continuous second order partial derivatives, show that $\text{curl}(\text{grad} f) = 0$

$\text{grad} f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$, and so $\text{curl}(\text{grad} f) = (\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}, \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}) = 0$ by equality of mixed partials.

(2) If $\overrightarrow{\mathbf{F}}$ is a vector fields in $\mathbb{R}^3$ with continuous second order partial differentials, prove that $\text{div}(\text{curl}\overrightarrow{\mathbf{F}}) = 0$

A similar straightfoward calculation.

8. (20 points) Given a vector field

$$\overrightarrow{\mathbf{F}}(x, y, z) = (y\cos z - yze^x, x\cos z - ze^x, -xy\sin z - ye^x).$$

Is it conservative? If so find a potential function $f(x, y)$.

If $\frac{\partial p}{\partial x} = y\cos z - yze^x$, then $p(x, y, z) = xy\cos z - yze^x + f(y, z)$. where f is some function of y and z alone. So if $\frac{\partial p}{\partial y} = x\cos z - ze^x + \frac{\partial f(y,z)}{\partial y} = x\cos z - ze^x$, then $f(y, z) = g(z)$, where g is a function of z alone. A similar argument shows $g(z) = C$, where C is a constant. It follows $p(x, y, z) = xy\cos z - yze^x + C$.

9. (20 points) Let S be the surface of the solid cylinder T bounded by $z = 0$, $z = 4$ and $x^2 + y^2 = 4$. Calculate the outward flux

$$\int\int_S \overrightarrow{\mathbf{F}} \cdot d\overrightarrow{S},$$

where

$$\overrightarrow{\mathbf{F}} = ((x^2 + y^2 + z^2)x, (x^2 + y^2 + z^2)y, (x^2 + y^2 + z^2)z)$$

First calculate $\text{div}(\overrightarrow{\mathbf{F}}) = 5(x^2 + y^2 + z^2)$. Then use Gauss's Theorem and cylindrical coordinates to get $\int\int_S \overrightarrow{\mathbf{F}} \cdot d\overrightarrow{S} = \int_0^{2\pi}\int_0^4\int_0^2 5r(r^2 + z^2)\, dr\, dz\, d\theta = \frac{1760\pi}{3}$.

10. (20 points)

(1) Let A be the area of the region bounded by a piecewise smooth simple closed curve C, show that

$$A = \oint_C xdy,$$

where C is oriented counter-clockwise.

Call the region D, then $A = \int\int_D dA = \oint_C xdy$ by Green's Theorem, where $Q(x, y) = x$ and $P(x, y) = 0$.

(2) Calculate the area of the region bounded by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Parameterize the ellipse by $x = a\cos t$, $y = b\sin t$. Then $dy = b\cos t\, dt$. Using part (a) we have $A = ab\int_0^{2\pi} cos^2 t\, dt = ab\pi$.